Position Domain Filtering and Range Domain Filtering for Carrier-Smoothed-Code DGNSS: An Analytical Comparison

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Abstract: Carrier-smoothed-code filters for differential Global Navigation Satellite System (GNSS) positioning can be partitioned into two groups: position domain filters and range domain filters. In carrier-smoothed-code filtering, noise terms of incremental carrier phase act as equivalent propagation noise. The equivalent propagation noises in carrier-smoothed-code filtering are temporally correlated and bounded in time, unlike white Gaussian noises in Kalman filtering. Thus, it seems that carrier-smoothed-code filtering does not inherit all the characteristics of classical Kalman filtering. To demonstrate that position domain filtering is better than range domain filtering a rigorous analysis is performed.

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1. Introduction

Due to the diversity of measurements, a Global Navigation Satellite System (GNSS) enables self-contained filtering for position estimates even under dynamic environments. The carrier-smoothed-code algorithm proposed by [1] maximally utilizes the information redundancy provided by GNSS. Here, the terms 'code' and 'carrier' denote pseudorange and carrier phase measurements, respectively. Compared with other types of filtering for precise differential positioning, carrier-smoothed-code filtering does not require assumed dynamic models or fast-rate aiding sensors for time propagation. Extending the range domain filter introduced by [1], position domain filters have been subsequently introduced. The three most representative among these are the complementary filter proposed by [2], the phase-connected filter proposed by [3], and the stepwise unbiased position projection filter (SUPF) proposed by [4].

In a carrier-smoothed-code filter, code measurements provide absolute position information while incremental carrier phase measurements provide displacement information. Thus, code noise acts as measurement noise and carrier noise acts as propagation noise. According to classical Kalman filtering theory, where all the noise terms are assumed white Gaussian, repetition of time propagation without measurement update would accumulate large estimation error without bound due to white Gaussian propagation noise [5]. In addition, position domain filtering (filtering in the state space) always shows better performance than range domain filtering (filtering in the measurement space) [6].

For differential carrier-smoothed-code filtering, incremental carrier phase measurements are prepared by differencing two instantaneous carrier phase measurements at successive epochs. Equivalent propagation noise in carrier-smoothed-code filtering is not white Gaussian since successive incremental carrier phase measurements share the same instantaneous carrier phase measurement [4]. Thus, basic insights obtained from Kalman filtering theory are not guaranteed to hold in the case of carrier-smoothed-code filtering. It is obvious that estimation error without measurement update by code measurements would not suffer accumulation of large error as long as the carrier phase signal is locked. Also, it is not clear that position domain filtering would still be preferred over range domain filtering in the case of carrier-smoothed-code filters.

The noise magnitudes of code and carrier phase measurements vary between different types of receivers. Furthermore, including the ionosphere-free, wide-lane, and narrow-lane combinations, various types of equivalent carrier phase measurements are available nowadays [7]. As can be easily verified, performance of any carrier-smoothed-code filter is largely affected by the magnitudes of the noise of the code and carrier phase measurements. Thus, when comparing the position domain and range domain filter algorithms, a rigorous analysis is more desirable than a limited number of simulations or experiments.

Motivated by the necessity for such an analytical comparison, this paper proposes an efficient analysis procedure to compare carrier-smoothed-code filters formulated both in the position domain and range domain. For the analysis, a stepwise optimal position projection filter (SOPF), a stepwise unbiased position projection filter (SUPF), and a stepwise optimal range projection filter (SORF) are utilized. Compared to other carrier-smoothed-code filters formulated in the position domain, the SOPF and SUPF introduced in [4] are advantageous in the context of performance analysis since they consider minimal number of states and provide consistent error covariance information. The SORF is a range domain carrier-smoothed filter that is based on the same stepwise minimization procedure as the SOPF and SUPF. Though specific filter algorithms are used, the analysis procedure presented here also provides an efficient methodology for obtaining tight upper bounds of position domain filters that are time-varying in nature due to changes in satellite geometry.

This paper is organized as follows. In Section II, the SOPF, SUPF, and SORF are summarized. In Section III, five theorems are analyzed with respect to the error covariance information provided by the position and range domain filters. Finally, concluding remarks will be given.

2. Carrier-smoothed-code filters with consistent error covariance

To extract the complete information transmitted by a GPS satellite, a receiver′s channel consists of two signal tracking loops, the Delay Lock Loop (DLL) and the Phase Lock Loop (PLL) [8]. The DLL is responsible for generating the pseudoranges, while the PLL generates the accumulated carrier phase observables. The pseudoranges and carrier phases measured by a single receiver are contaminated by a variety of error sources, including receiver clock bias, thermal noise, satellite clock bias, ionospheric delay, tropospheric delay, and multipath disturbance. If a user′s receiver and a reference receiver are located close by, the common-mode error sources such as the satellite clock bias, ionospheric delay, and tropospheric delay can be effectively eliminated [7-9]. This type of data combination is referred to as *single-differencing*. The multipath error can be effectively detected and mitigated by various other methods [10-14]. The corrected pseudorange $\tilde{\rho}_{j,k}$ and carrier phase $\tilde{\phi}_{j,k}$ (assuming the common-mode errors and multipath error have been eliminated) can be modelled as [7-9]:

$$\tilde{\rho}_{j,k} = e_{j,k}^T (x_{j,k} - x_{u,k}) + b_{u,k} + v_{j,k}$$

$$\tilde{\phi}_{j,k} = e_{j,k}^T (x_{j,k} - x_{u,k}) + b_{u,k} + n_{j,k} + \lambda N_j \tag{1}$$

$$\begin{bmatrix} v_{j,k} \\ v_{j,k+1} \\ v_{j+1,k} \\ n_{j,k} \\ n_{j,k+1} \\ n_{j+1,k} \end{bmatrix} \sim \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} r_\rho & 0 & 0 & 0 & 0 & 0 \\ 0 & r_\rho & 0 & 0 & 0 & 0 \\ 0 & 0 & r_\rho & 0 & 0 & 0 \\ 0 & 0 & 0 & r_\Phi & 0 & 0 \\ 0 & 0 & 0 & 0 & r_\Phi & 0 \\ 0 & 0 & 0 & 0 & 0 & r_\Phi \end{bmatrix} \right)$$

where

$e_{j,k}$: Line Of Sight (LOS) vector from the receiver to the j -th satellite

$x_{j,k}$: Earth-Centred Earth-Fixed (ECEF) position of the j -th satellite

$x_{u,k}$: ECEF receiver position

$b_{u,k}$: receiver clock bias

N_j : unresolved integer ambiguity

$v_{j,k}$, $n_{j,k}$: noise terms in the code and carrier measurements

r_ρ, r_ϕ : uniform noise variances of code and carrier measurements

$X \sim (m, P)$: Gaussian random vector with the mean of m and the covariance of P

The symbol X_k will be used to denote true state vector that is composed of the three-dimensional position sub-vector $x_{u,k}$ and the clock bias $b_{u,k}$:

$$X_k := \begin{bmatrix} x_{u,k} \\ \cdots \\ b_{u,k} \end{bmatrix} \tag{2}$$

Related to the true state vector X_k, the symbols $\overline{X}_k$, $\delta\overline{X}_k$, $\overline{P}_k$, $\hat{X}_k$, $\delta\hat{X}_k$, and $\hat{P}_k$ will be used to represent the *a priori* state estimate, *a priori* estimation error, *a priori* error covariance matrix, *a posteriori* state estimate, *a posteriori* estimation error, and *a posteriori* error covariance matrix at the k-th step for all the three filters, respectively.

$$\overline{X}_k = \begin{bmatrix} \overline{x}_{u,k} \\ \cdots \\ \overline{b}_{u,k} \end{bmatrix}, \quad \delta\overline{X}_k := \overline{X}_k - X_k = \begin{bmatrix} \delta\overline{x}_{u,k} \\ \cdots \\ \delta\overline{b}_{u,k} \end{bmatrix}, \quad \delta\overline{X}_k \sim (O, \overline{P}_k)$$

$$\hat{X}_k = \begin{bmatrix} \hat{x}_{u,k} \\ \cdots \\ \hat{b}_{u,k} \end{bmatrix}, \quad \delta\hat{X}_k := \hat{X}_k - X_k = \begin{bmatrix} \delta\hat{x}_{u,k} \\ \cdots \\ \delta\hat{b}_{u,k} \end{bmatrix}, \quad \delta\hat{X}_k \sim (O, \hat{P}_k) \tag{3}$$

To denote the true displacement information, the symbol ΔX_k will be used:

$$\Delta X_k := X_{k+1} - X_k = [(\Delta x_{u,k})^T \vdots \Delta b_{u,k}]^T \tag{4}$$

If it is necessary to discriminate the variables of the three different filters, the superscripts o, s, and r are used to denote the SOPF, SUPF, and SORF, respectively.

2.2 Position Domain Filters

In the position domain carrier-smoothed-code filters, all the channelwise scalar measurements participate in position estimation concurrently, as they are acquired. Thus, vector form notation is more convenient in describing position domain filters. Adopting vector notations, the indirect measurement vector Z_k to update the position estimate $\overline{X}_k$ from to $\hat{X}_k$ is written as follows:

$$Z_k = H_k \delta\overline{X}_k + v_k, \quad v_k \sim (O_{J\times 1}, r_\rho I_{J\times J}) \tag{5}$$

where

$$Z_k := \begin{bmatrix} z_{1,k} \\ z_{2,k} \\ \vdots \\ z_{J,k} \end{bmatrix}, \quad H_k := \begin{bmatrix} h_{1,k} \\ h_{2,k} \\ \vdots \\ h_{J,k} \end{bmatrix}, \quad v_k := \begin{bmatrix} v_{1,k} \\ v_{2,k} \\ \vdots \\ v_{J,k} \end{bmatrix}$$

$$z_{j,k} := \tilde{\rho}_{j,k} - e_{j,k}^T (x_{j,k} - \overline{x}_{u,k}) - \overline{b}_{u,k} = h_{j,k} \delta\overline{X}_k + v_{j,k}$$

$$h_{j,k} := [e_{j,k}^T \;\vdots\; -1]$$

$O_{J\times 1}$: $J \times 1$ zero vector

$I_{J\times J}$: $J \times J$ identity matrix

J : number of visible satellites (6)

Similarly, the indirect measurement vector Ω_{k+1} to propagate $\hat{X}_k$ in time toward $\overline{X}_{k+1}$ is written as follows:

$$\Omega_{k+1} = H_{k+1} \Delta X_k + W_{k+1}$$

$$W_{k+1} = -\Delta H_k \delta\hat{X}_k - n_{k+1} + n_k, \quad n_k \sim (O_{J\times 1}, r_\Phi I_{J\times J}) \tag{7}$$

where

$$\Omega_{k+1} := \begin{bmatrix} \omega_{1,k+1} \\ \omega_{2,k+1} \\ \vdots \\ \omega_{J,k+1} \end{bmatrix}, \quad W_{k+1} := \begin{bmatrix} w_{1,k+1} \\ w_{2,k+1} \\ \vdots \\ w_{J,k+1} \end{bmatrix}, \quad n_k := \begin{bmatrix} n_{1,k} \\ n_{2,k} \\ \vdots \\ n_{J,k} \end{bmatrix}$$

$$\Delta H_k := H_{k+1} - H_k$$

$$\omega_{j,k+1} := e_{j,k}^T \Delta x_{j,k} + \Delta e_{j,k}^T (x_{j,k+1} - \hat{x}_{u,k}) - (\tilde{\phi}_{j,k+1} - \tilde{\phi}_{j,k})$$
$$= h_{j,k+1} \Delta X_k + w_{j,k+1}$$

$$w_{j,k+1} := -\Delta e_{j,k}^T \delta\hat{x}_{u,k} - n_{j,k+1} + n_{j,k} \tag{8}$$

By applying a stepwise minimization procedure [4] to the measurement vectors Z_k and Ω_{k+1} for measurement update and time propagation, respectively, the SOPF algorithm is obtained as follows.

Initialization:

$$\hat{X}_{k0}^o = E[X_{k0} \mid \tilde{\rho}_{k0}]$$

$$\hat{P}_{k0}^o = r_\rho [H_{k0}^T H_{k0}]^{-1} \tag{9}$$

Time-Propagation:

$$Q_{k+1} = \Delta H_{k+1} \hat{P}_k^o (\Delta H_{k+1})^T + 2 r_\Phi I_{J \times J}$$
$$+ r_\Phi \Delta H_{k+1} (I_{4\times4} - K_k^o H_k) U_k^o$$
$$+ r_\Phi (U_k^o)^T (I_{4\times4} - K_k^o H_k)^T (\Delta H_{k+1})^T$$

$$U_{k+1}^o = [(H_{k+1})^T (Q_{k+1})^{-1} H_{k+1}]^{-1} (H_{k+1})^T (Q_{k+1})^{-1}$$

$$\overline{X}_{k+1}^o = \hat{X}_k^o + U_{k+1}^o \Omega_{k+1}$$

$$\overline{P}_{k+1}^o = U_{k+1}^o \left\{ H_k \hat{P}_k^o H_k^T + r_\Phi \begin{bmatrix} 2 I_{J\times J} - H_k \left(I_{4\times4} - K_k^o H_k\right) U_k^o \\ -(U_k^o)^T \left(I_{4\times4} - K_k^o H_k\right)^T (H_k)^T \end{bmatrix} \right\} (U_{k+1}^o)^T \tag{10}$$

Measurement Update:

$$K_k^o = \overline{P}_k^o (H_k)^T [H_k \overline{P}_k^o (H_k)^T + r_\rho I_{J\times J}]^{-1}$$

$$\hat{X}_k^o = \overline{X}_k^o - K_k^o Z_k$$

$$\hat{P}_k^o = \left(I_{4\times4} - K_k^o H_k\right) \overline{P}_k^o \left(I_{4\times4} - K_k^o H_k\right)^T + r_\rho K_k^o (K_k^o)^T \tag{11}$$

If computational efficiency is desired over theoretical accuracy, the SUPF can replace the SOPF. The main difference between the SOPF and SUPF is in the formulation of the gain matrix with respect to time propagation. The SUPF algorithm is summarized as follows.

Initialization:

$$\hat{X}^s_{k0} = E[X_{k0} \mid \tilde{\rho}_{k0}]$$

$$\hat{P}^s_{k0} = r_\rho [H^T_{k0} H_{k0}]^{-1} \tag{12}$$

Time-Propagation:

$$U^s_{k+1} = [H^T_{k+1} H_{k+1}]^{-1} H^T_{k+1}$$

$$\overline{X}^s_{k+1} = \hat{X}^s_k + U^s_{k+1} \Omega_{k+1}$$

$$\overline{P}^s_{k+1} = U^s_{k+1} \left[\left(1 + 2r_\Phi / r_\rho\right) H_k \hat{P}^s_k H^T_k - 2r_\Phi H_k (H^T_k H_k)^{-1} H^T_k + 2r_\Phi I_{J\times J} \right] (U^s_{k+1})^T \tag{13}$$

Measurement Update:

$$K^s_k = \overline{P}^s_k (H_k)^T [H_k \overline{P}^s_k (H_k)^T + r_\rho I_{J\times J}]^{-1}$$

$$\hat{X}^s_k = \overline{X}^s_k - K^s_k Z_k$$

$$\hat{P}^s_k = \left(I_{4\times4} - K^s_k H_k\right) \overline{P}^s_k \left(I_{4\times4} - K^s_k H_k\right)^T + r_\rho K^s_k (K^s_k)^T \tag{14}$$

In both the SOPF and SUPF, the estimation error changes according to the following recursive relations:

$$\delta\overline{X}_{k+1} = U_{k+1} [H_k \delta\hat{X}_k - (n_{k+1} - n_k)]$$

$$\delta\hat{X}_k = \left(I_{4\times4} - K_k H_k\right) \delta\overline{X}_k - K_k v_k \tag{15}$$

By the information sharing principle [15], the error covariance update shown in Eqs. (11) and (14) are commonly replaced by the following equation (where superscripts are omitted for clarity):

$$(\hat{P}_k)^{-1} = (\overline{P}_k)^{-1} + \frac{1}{r_\rho} (H^T_k H_k)^{-1} \tag{16}$$

2.2 Range Domain Filter

The SORF generates two range estimates: a compressed pseudorange $\hat{\rho}_{j,k}$ and a projected pseudorange $\bar{\rho}_{j,k}$. The compressed pseudorange $\hat{\rho}_{j,k}$ is the *a posteriori* range estimate that is based on all the code and carrier measurements $\{\tilde{\rho}_{j,i}\}_{i=0,1,2,\cdots,k}$ and $\{\phi_{j,i}\}_{i=0,1,2,\cdots,k}$ after the initialization of the filter. The projected pseudorange $\bar{\rho}_{j,k}$ is the *a priori* range estimate before the new pseudorange measurement $\tilde{\rho}_{j,k}$ at the current k −th step is accommodated. The error covariance values of the compressed pseudorange $\hat{\rho}_{j,k}$ and the projected pseudorange $\bar{\rho}_{j,k}$ are denoted by $\hat{R}_{j,k}$ and $\bar{R}_{j,k}$, respectively.

Compared to the SOPF and SUPF, the SORF needs no permanent storage of the position estimate. Instead, the position estimates are generated at each time by treating the outputs of multiple range domain filters $\{\hat{\rho}_{j,k}\}_{j=1,2,\cdots,J}$ or $\{\bar{\rho}_{j,k}\}_{j=1,2,\cdots,J}$ in the same manner as that applied to the raw pseudoranges $\{\tilde{\rho}_{j,k}\}_{j=1,2,\cdots,J}$ for each k. Since the equivalent pseudoranges of different channels are not correlated to each other, the following error covariance matrix can represent the accuracy of the resultant position estimate:

$$\bar{P}_k^r = (H_k^T \bar{R}_k^{-1} H_k)^{-1}$$

$$\hat{P}_k^r = (H_k^T \hat{R}_k^{-1} H_k)^{-1} \tag{17}$$

$\bar{R}_k$ and $\hat{R}_k$ in Eq. (17) are constructed at each time step by utilizing the scalar error covariance values of all the channels:

$$\bar{R}_k := \begin{bmatrix} \bar{R}_{1,k} & 0 & \cdots & 0 \\ 0 & \bar{R}_{2,k} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{R}_{J,k} \end{bmatrix}, \quad \hat{R}_k := \begin{bmatrix} \hat{R}_{1,k} & 0 & \cdots & 0 \\ 0 & \hat{R}_{2,k} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{R}_{J,k} \end{bmatrix} \tag{18}$$

With an understanding of the above subtle differences between range domain filtering and position domain filtering, the SORF algorithm can be easily derived by applying the same stepwise minimization procedire that was used to derive the SOPF and SUPF [4]. The resultant SORF algorithm is summarized as follows.

Initialization:

$$\alpha_{j,k0} = 0,\ \ \beta_{j,k0} = 1,\ \ \hat{\rho}_{j,k0} = \tilde{\rho}_{j,k0},\ \ \hat{R}_{j,k0} = r_{\rho} \tag{19}$$

Time-Propagation:

$$\bar{\rho}_{j,k+1} := \hat{\rho}_{j,k} + (\tilde{\phi}_{j,k+1} - \tilde{\phi}_{j,k})$$

$$\bar{R}_{j,k+1} = \hat{R}_{j,k} + 2\beta_{j,k} r_{\Phi} \tag{20}$$

Measurement Update:

$$\alpha_{j,k} = r_{\rho} / (\bar{R}_{j,k} + r_{\rho})$$

$$\beta_{j,k} = 1 - \alpha_{j,k}$$

$$\hat{\rho}_{j,k} = \alpha_{j,k}\bar{\rho}_{j,k} + \beta_{j,k}\tilde{\rho}_{j,k}$$

$$\hat{R}_{j,k} = \alpha_{j,k}\bar{R}_{j,k} \tag{21}$$

Position Solutions:

$$\bar{X}_k^r = E[X_k \mid \bar{\rho}_k],\ \ \bar{P}_{k0} = \bar{R}_k (H_k^T H_k)^{-1}$$

$$\hat{X}_k^r = E[X_k \mid \hat{\rho}_k],\ \ \hat{P}_{k0} = \hat{R}_k (H_k^T H_k)^{-1} \tag{22}$$

3. Analytical comparison of position domain and range domain filters

As shown in the previous section, the position domain filters SOPF and SUPF, compared to the range domain filter SORF, are time-varying due to the satellites′ changing geometry. Thus, a theoretically rigorous analysis of these filters is, in general, more difficult. To overcome this difficulty, five theorems are introduced. Theorem 1 provides a uniform lower bound for the SOPF. Theorem 2 provides an apparent inequality between the SOPF and SUPF. Theorem 3 and Theorem 4 together show that the SUPT is upper bounded by the SORF. Finally, Theorem 5 provides an asymptotic error covariance limit of the SORF.

Theorem 1: Lower bound for error covariance of SOPF

The SOPF′s error covariance matrix $\hat{P}_k^o$ is lower bounded by the following matrix inequality for all k:

$$\frac{r_\Phi r_\rho}{r_\Phi + r_\rho}(H_k^T H_k)^{-1} \le \hat{P}_k^o \tag{23}$$

<Proof>

It is obvious that the carrier noise n_{k+1} is not correlated with the carrier noise n_k and the *a posteriori* estimation error $\delta\hat{X}_k^o$. Thus, the following inequality holds:

$$\begin{aligned}\overline{P}_{k+1}^o &= U_{k+1}^o E\left\{[(H_k \delta\hat{X}_k^o + n_k) - n_{k+1}][(H_k \delta\hat{X}_k^o + n_k) - n_{k+1}]^T\right\}(U_{k+1}^o)^T \\ &\ge r_\Phi U_{k+1}^o (U_{k+1}^o)^T \\ &= r_\Phi [H_{k+1}^T (Q_{k+1}^o)^{-1} H_{k+1}]^{-1} H_{k+1}^T (Q_{k+1}^o)^{-1} I_{J\times J} (Q_{k+1}^o)^{-1} H_{k+1} [H_{k+1}^T (Q_{k+1}^o)^{-1} H_{k+1}]^{-1}\end{aligned} \tag{24}$$

Due to the characteristics of projection matrices [16], the following matrix inequality holds for any time step k and for any satellite geometry:

$$H_k (H_k^T H_k)^{-1} H_k^T \quad \le \quad I_{J\times J} \tag{25}$$

Substituting Eq. (25) into Eq. (24):

$$\overline{P}_{k+1}^o \ge r_\Phi (H_{k+1}^T H_{k+1})^{-1} \tag{26}$$

Applying Eq. (26) to Eq. (16), it can be shown that the *a posteriori* error covariance matrix satisfies the condition:

$$(\hat{P}^{o}_{k+1})^{-1} = \frac{1}{r_\Phi} H^T_{k+1} H_{k+1} + \frac{1}{r_\rho} H^T_{k+1} H_{k+1} \leq \frac{r_\Phi + r_\rho}{r_\Phi r_\rho} H^T_{k+1} H_{k+1} \tag{27}$$

Taking the inverse of both sides of Eq. (27), Eq. (23) is obtained. □

Theorem 2: Inequality between error covariance matrices of SOPF and SUPF

The *a priori* error and *a posteriori* error covariance matrices $\overline{P}^o_k$, $\hat{P}^o_k$, $\overline{P}^s_k$, and $\hat{P}^s_k$ of the SOPF and SUPF satisfy the following matrix inequalities for all k :

$$O < \overline{P}^o_k \leq \overline{P}^s_k , \qquad O < \hat{P}^o_k \leq \hat{P}^s_k \tag{28}$$

□

Remarks: The proof of Theorem 3 is omitted since it is obvious. Interested readers can refer to [4].

Lemma: Boundedness and equivalence of scalar covariance recursion

Consider the following scalar covariance recursions of $\overline{\mu}_k$ and $\hat{\mu}_k$:

$$\hat{\mu}_0 = r_\rho \tag{29}$$

$$\overline{\mu}_{k+1} = \left(1 + 2 r_\Phi / r_\rho\right)\hat{\mu}_k \tag{30}$$

$$\hat{\mu}_k = \frac{r_\rho}{r_\rho + \overline{\mu}_k}\overline{\mu}_k \tag{31}$$

Then, Eqs. (30) and (31) are equivalent to the following equations:

$$\overline{\mu}_{k+1} = \hat{\mu}_k + 2\beta^\mu_k r_\Phi , \qquad \beta^\mu_k := \frac{\overline{\mu}_k}{\overline{\mu}_k + r_\rho} \tag{32}$$

$$\hat{\mu}^{-1}_k = \overline{\mu}^{-1}_k + r^{-1}_\rho \tag{33}$$

In addition, $\overline{\mu}_k$ and $\hat{\mu}_k$ are non-increasing and bounded below for all k and converge to the limit values $\overline{\mu}$ and $\hat{\mu}$ as follows:

$$\overline{\mu}_k \geq 2 r_\Phi , \qquad \lim_{k\to\infty} \overline{\mu}_k = \overline{\mu} , \qquad \overline{\mu} := 2 r_\Phi \tag{34}$$

$$\hat{\mu}_k \geq \frac{2r_\Phi r_\rho}{2r_\Phi + r_\rho}, \quad \lim_{k\to\infty} \hat{\mu}_k = \hat{\mu}, \quad \hat{\mu} := 2r_\Phi \tag{35}$$

<Proof>

Comparing Eq. (31) and (33), their equivalence can be easily found. From now on, it is shown that Eq. (30) and Eq. (32) are equivalent. Rewriting Eq. (32), the following equations are obtained:

$$\beta_k^\mu (\overline{\mu}_k + r_\rho) = \overline{\mu}_k$$

$$\overline{\mu}_k = \frac{\beta_k^\mu}{1-\beta_k^\mu} r_\rho \tag{36}$$

Combining Eq. (36) with Eq. (33) yields:

$$\beta_k^\mu = \frac{\hat{\mu}_k}{r_\rho} \tag{37}$$

Substituting $\beta_{l,k}$ in Eq. (37) into Eq. (32) again yields Eq. (30).

From now on, it will be shown that $\{\overline{\mu}_k\}_{k=1,2,3,\cdots}$ is non-increasing i.e.,

$$2r_\Phi \leq \overline{\mu}_{k+1} \leq \overline{\mu}_k \,. \tag{38}$$

At first, by utilizing Eqs. (29) and (30), it can be shown that Eq. (38) is satisfied at $k=1$. Next, assume that $\overline{\mu}_k$ satisfies the following equation:

$$\overline{\mu}_k = 2r_\Phi + \varepsilon_k, \quad \varepsilon_k \geq 0 \tag{39}$$

To show that Eq. (39) is also satisfied at the next time step, the recursion of $\{\overline{\mu}_k\}_{k=1,2,3,\cdots}$ is derived as follows by combining Eqs. (30) and (31):

$$\overline{\mu}_{k+1} = \frac{r_\rho + 2r_\Phi}{r_\rho + \overline{\mu}_k} \overline{\mu}_k \tag{40}$$

Substituting Eq. (39) to Eq. (40) yields:

$$\overline{\mu}_{k+1} = 2r_\Phi + \varepsilon_{k+1} \tag{41}$$

$$\varepsilon_{k+1} = \frac{r_\rho}{r_\rho + 2r_\Phi + \varepsilon_k} \varepsilon_k \geq 0 \tag{42}$$

Thus, Eq. (39) which is equivalent to the inequality of Eq. (34) is satisfied. Combining Eq. (33)

and the inequality of Eq. (34), the inequality of Eq. (35) is obtained.

By observing Eq. (42), it can be found that:

$$0 < \frac{r_\rho}{r_\rho + 2r_\Phi + \varepsilon_k} < 1 \qquad \text{for all } k, \tag{43}$$

which means that

$$\lim_{k\to\infty} \varepsilon_k = 0. \tag{44}$$

Combining Eqs. (39) and (44), the limit value $\bar{\mu}$ is obtained. Substituting $\bar{\mu}$ to Eq. (31), the other limit value $\hat{\mu}$ is obtained. □

Theorem 3: Upper bound for error covariance of SUPF

The *a priori* error and *a posteriori* error covariance matrices $\bar{P}_k^s$ and $\hat{P}_k^s$ of the SUPF is upper bounded by the following matrix inequalities for all k:

$$\bar{P}_k^s \le \bar{\mu}_k (H_k^T H_k)^{-1} \tag{45}$$

$$\hat{P}_k^s \le \hat{\mu}_k (H_k^T H_k)^{-1} \tag{46}$$

where the positive real values $\bar{\mu}_k$ and $\hat{\mu}_k$ are generated by Eqs. (29-31).

<Proof>

It is obvious that at the initial time step $k = 0$, Eq. (46) is satisfied. Next, assume that $\hat{P}_k^s$ satisfies Eq. (46). Then, $\bar{P}_{k+1}^s$ (shown in Eq. (13)) satisfies the following matrix inequality according to Eq. (46):

$$\begin{aligned} \bar{P}_{k+1}^s &\le U_{k+1}^s \left\{ \begin{aligned} &\left(1 + 2r_\Phi / r_\rho\right)\hat{\mu}_k H_k (H_k^T H_k)^{-1} H_k^T \\ &+ 2r_\Phi I - 2r_\Phi H_k (H_k^T H_k)^{-1} H_k^T \,] \end{aligned} \right\} (U_{k+1}^s)^T \\ &= U_{k+1}^s \left\{ \left[(1 + 2r_\Phi / r_\rho)\hat{\mu}_k - 2r_\Phi \right] H_k (H_k^T H_k)^{-1} H_k^T + 2r_\Phi I \right\} (U_{k+1}^s)^T \end{aligned} \tag{47}$$

According to Eq. (35), $\hat{\mu}_k$ in Eq. (47) satisfies the condition:

$$(1 + 2r_\Phi / r_\rho)\hat{\mu}_k - 2r_\Phi \ge 0. \tag{48}$$

Applying Eqs. (25) and (48) to Eq. (47), the following inequality is obtained:

$$
\begin{aligned}
\overline{P}^{s}_{k+1} &\le [(1+2r_{\Phi}/r_{\rho})\hat{\mu}_k - 2r_{\Phi}]U^{s}_{k+1}[H_k(H_k^T H_k)^{-1}H_k^T](U^{s}_{k+1})^T + 2r_{\Phi}U^{s}_{k+1}(U^{s}_{k+1})^T \\
&\le [(1+2r_{\Phi}/r_{\rho})\hat{\mu}_k - 2r_{\Phi}]U^{s}_{k+1}(U^{s}_{k+1})^T + 2r_{\Phi}U^{s}_{k+1}(U^{s}_{k+1})^T \\
&= \left(1+2r_{\Phi}/r_{\rho}\right)\hat{\mu}_k U^{s}_{k+1}(U^{s}_{k+1})^T \\
&= \overline{\mu}_{k+1}(H_{k+1}^T H_{k+1})^{-1}
\end{aligned}
\quad (49)
$$

Thus, it can be seen that Eq. (45) is satisfied. Finally, assume that $\overline{P}^{s}_{k}$ satisfies Eq. (45), then $\hat{P}^{s}_{k}$ satisfies the following matrix inequality by applying Eq. (45) to the information sharing principle in Eq. (16):

$$
(\hat{P}^{s}_{k})^{-1} \ge (1/\overline{\mu}_k)H_k^T H_k + (1/r_{\rho})H_k^T H_k = (1/\overline{\mu}_k + 1/r_{\rho})H_k^T H_k \quad (50)
$$

By taking the inverse of both sides of Eq. (50), Eq. (46) is verified. □

Theorem 4: Geometry-free upper bound of SUPF by SORF

The *a priori* error and *a posteriori* error covariance matrices $\overline{P}^{s}_{k}$ and $\hat{P}^{s}_{k}$ of the SUPF are upper bounded by the *a priori* error and *a posteriori* error covariance matrices $\overline{P}^{r}_{k}$ and $\hat{P}^{r}_{k}$ of the SORF for all k, respectively:

$$
\overline{P}^{s}_{k} \le \overline{P}^{r}_{k}, \qquad \hat{P}^{s}_{k} \le \hat{P}^{r}_{k} \quad (51)
$$

<Proof>

Assume that the l-th satellite′s signal has been locked during the longest period among the J visible satellites at the k-th time step. Then the error covariance values $\{\overline{R}_{j,k}\}_{j=1,2,\cdots,J}$ and $\{\hat{R}_{j,k}\}_{j=1,2,\cdots,J}$ with respect to all the visible satellites are lower bounded by the following inequalities:

$$
\overline{R}_{j,k} \ge \overline{R}_{l,k}, \qquad \hat{R}_{j,k} \ge \hat{R}_{l,k}, \qquad j = 1, 2, \cdots, J \quad (52)
$$

By combining Eqs. (17), (18), and (52), it can be shown that the error covariance matrices by combining the range domain filter is lower bounded by the following inequalities:

$$
\overline{P}^{r}_{k} \ge \overline{R}^{l}_{k} \cdot \left(H_k^T H_k\right)^{-1}, \qquad \hat{P}^{r}_{k} \ge \hat{R}^{l}_{k} \cdot \left(H_k^T H_k\right)^{-1} \quad (53)
$$

As shown in Eq. (53), $\overline{P}_k^r$ and $\hat{P}_k^r$ are lower bounded by two factors. One is the geometry-dependent factor and the other is the geometry-free factor. The geometry-dependent factor is due to $\left(H_k^T H_k\right)^{-1}$ and the geometry-free factor is the result of the scalar error covariance values $\overline{R}_k^l$ and $\hat{R}_k^l$. By comparing Eqs. (19-21) with Eqs. (29-31), it can found that $\{\overline{R}_{l,k}, \hat{R}_{l,k}\}$ and $\{\overline{\mu}_k, \hat{\mu}_k\}$ are generated by the same algorithm. However, due to different susceptibilities of information loss by intermittent satellite outages, the following inequalities hold for all k:

$$\overline{R}_k^l \geq \overline{\mu}_k, \qquad \hat{R}_k^l \geq \hat{\mu}_k \tag{54}$$

Combining Eqs. (45), (46), (53), and (54), the proof is completed. □

Theorem 5: Asymptotic error covariance values of SORF

The scalar error covariance values $\overline{R}_{j,k}$ and $\hat{R}_{j,k}$ of the SORF asymptotically converge to $\overline{R}$ and $\hat{R}$, respectively:

$$\lim_{k\to\infty} \overline{R}_{j,k} = \overline{R}, \quad \lim_{k\to\infty} \hat{R}_{j,k} = \hat{R} \qquad \text{for all} \quad j = 1,2,\cdots,J \tag{55}$$

where

$$\hat{R} = \frac{2r_\Phi r_\rho}{2r_\Phi + r_\rho}, \quad \overline{R} = 2r_\Phi \tag{56}$$

<Proof>

Comparison between Eqs. (19-21) and Eqs. (29-31) reveals that $\{\overline{R}_{l,k}, \hat{R}_{l,k}\}$ and $\{\overline{\mu}_k, \hat{\mu}_k\}$ are generated by the same algorithm. By applying the same procedure that was applied to $\{\overline{\mu}_k, \hat{\mu}_k\}$, Eqs. (55) and (56) are obtained.

Corollary: Asymptotic performance bounds

If the visible satellite set does not change, the error covariance matrices of the three proposed filters satisfy the following matrix inequality for all $k \geq k_s$ after the geometry-free error factor enters into a steady states condition:

$$\frac{r_\Phi r_\rho}{r_\Phi + r_\rho}(H_k^T H_k)^{-1} \le \hat{P}_k^o \le \hat{P}_k^s \le \hat{P}_k^r \le \frac{2r_\Phi r_\rho}{2r_\Phi + r_\rho}(H_k^T H_k)^{-1} \quad (57)$$

Remarks: As a direct consequence of Theorem 1, Theorem 2, Theorem 3, Theorem 4, and Theorem 5, Eq. (57) shows that the position domain filters SOPF and SUPF, in general, provide more accurate position estimates than the range domain filter SORF. However, Eq. (57) also shows that the steady state performance of all the three filters are very similar.

4. Conclusion

This paper has proposed a theoretically rigorous analysis procedure that compares the position domain and range domain carrier-smoothed-code filters for differential GNSS positioning. The analysis is performed utilizing the consistent error covariance matrices of the SOPF, SUPF, and SORF as performance measures. In spite of peculiar noise characteristics that occur in propagating errors in time, it has been shown that filtering in the position domain is, in theory, more advantageous than range domain carrier-smoothed-code filtering. This is also the case for classical Kalman filtering. However, it was also shown that if the visible satellite set does not change during a sufficiently long time interval, the performance of all three filters is similar.

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6. References

1 HATCH, R.R.: 'The Synergism of GPS Code and Carrier Measurements'. *Proceedings of the Third International Geodetic Symposium on Satellite Doppler Positioning, Vol. II*, February 1982, New Mexico, pp. 1213-1232.

2 HWANG, P.Y.C., AND BROWN, R.G..: 'GPS Navigation: Combining Pseudorange with Continuous Carrier Phase Using a Kalman Filter'. *Navigation: Journal of The Institute of Navigation,* 1990, **37** (2) pp. 181-196

3 BISNATH, S.B., AND LANGLEY, R.B.: 'Precise, Efficient GPS-Based Geometric Tracking of Low Earth Orbiters'. *Proceedings of the Institute of Navigation Annual Meeting*, June 1999, Cambridge, Massachusetts, pp. 751-760

4 LEE, H.K., RIZOS, C., AND JEE, G.I.: 'Design of Kinematic DGPS Filters with Consistent Error Covariance Information', submitted to *IEE Proceedings-Radar, Sonar and Navigation* for possible publication, January, 2003

5 Jazwinski, A.H.: '*Stochastic Processes and Filtering Theory*' (Academic Press, 1970.)

6 LEE, H.K.: 'Time-Propagated Measurement Fusion and Its Application to Multipath Detection and Isolation'. PhD Dissertation, School of Electrical Engineering and Computer Science, Seoul National University, 2002

7 Hoffmann-Wellenhof, B., Lichtenegger, H. and Collins, J.: '*GPS Theory and Practice*' (Springer-Verlag, 1994 4th edn.)

8 PARKINSON, B., AND AXELAD, P.: '*Global Positioning System: Theory and Applications*' (American Institute of Aeronautics and Astronautics, 1996.)

9 FARRELL, J.A., AND BATH, M.: '*The Global Positioning System and Inertial Navigation*' (McGraw-Hill, 1998.)

10 VAN NEE, R.D.J., 'The Multipath Estimating Delay Lock Loop: Approaching Theoretical Accuracy Limits', *Proceedings of Position Location and Navigation Symposium*, April 1994, Las Vegas, Nevada, pp. 246 –251.

11 BRAASCH, M.S., 'GPS Multipath Model Validation', *Proceedings of Position Location and Navigation Symposium*, April 1996, Atlanta, GA, pp. 672-678.

12 AXELRAD, P., COMP, C.J., AND MACDORAN, P.F., 'SNR-Based Multipath Error Correction for GPS Differential Phase', *IEEE Tr. on Aerospace and Electronic Systems*, 1996, **32** (2) pp. 650-660.

13 GEORGIADOU, Y., AND KLEUSBERG, A., ‛On Carrier Signal Multipath Effects in Relative GPS Positioning′, *Manuscripta Geodaetica*, 1988, **13** (1) pp. 1-8.

14 RAY, J.K., CANNON, M.E., AND FENTON, P., ‛GPS code and carrier multipath mitigation using a multiantenna system′, *IEEE Tr. on Aerospace and Electronic System*s, 2001, **37** (1) pp. 183-195

15 CARLSON, N.A.: ‛Federated Square Root Filter for Decentralized Parallel Processes′, *IEEE Transactions on Aerospace and Electronic Systems*, 1990, **26** (3) pp. 517-525.

16 KAILATH, T., SAYED, A.H., AND HASSIBI, B.: ‛*Linear Estimation*′ (Prentice Hall, 2000.)