ETRI 원내 전문 교육

GPS/관성센서 통합에 의한 측위 및 응용

LEC1 INS FUNDAMENTALS

2005/7/14

이 형 근 (hyknlee@hau.ac.kr)

항법 및 정보시스템 연구실
Navigation & Information Systems Laboratory

항공전자 및 정보통신공학부

한국항공대학교
HANKUK AVIATION UNIVERSITY

Navigation ?

- **To accurately determine position and velocity relative to a known reference**

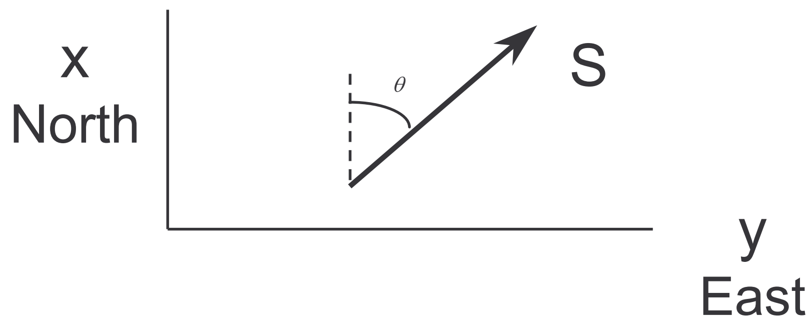
- **To plan and execute the maneuvers necessary to move between desired locations**

Classification of Nav. Systems

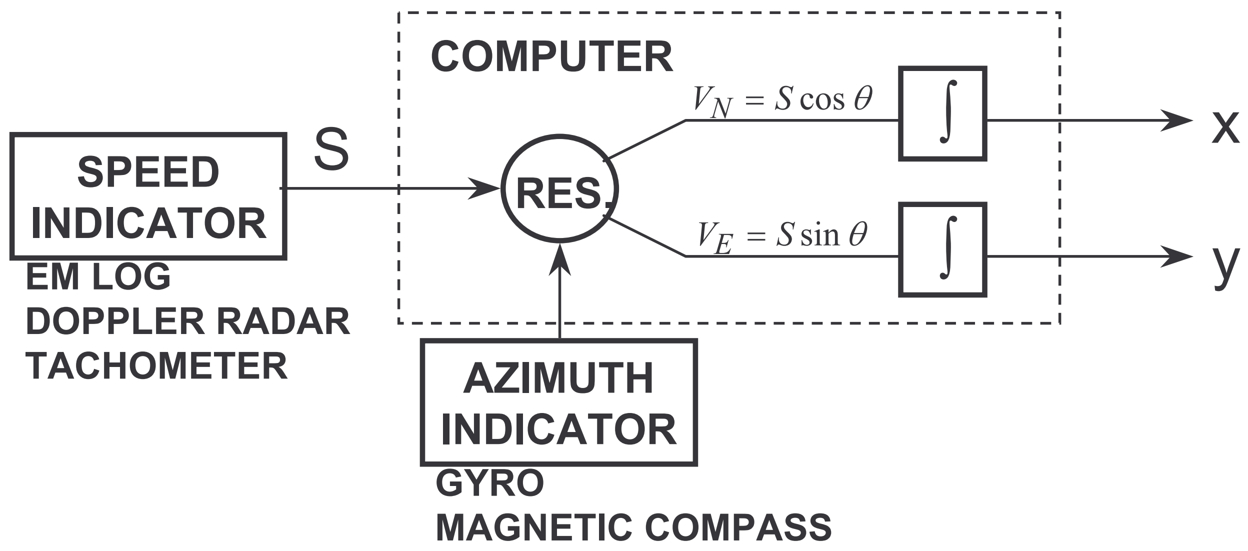
- Radio Navigation Systems
  - GPS, Galilero
  - VOR/DME, ILS, TACAN, Loran
- Dead reckoning systems
  - Inertial Navigation Systems (INS)
  - Gyro/tachometer systems: Land applications

Dead Reckoning

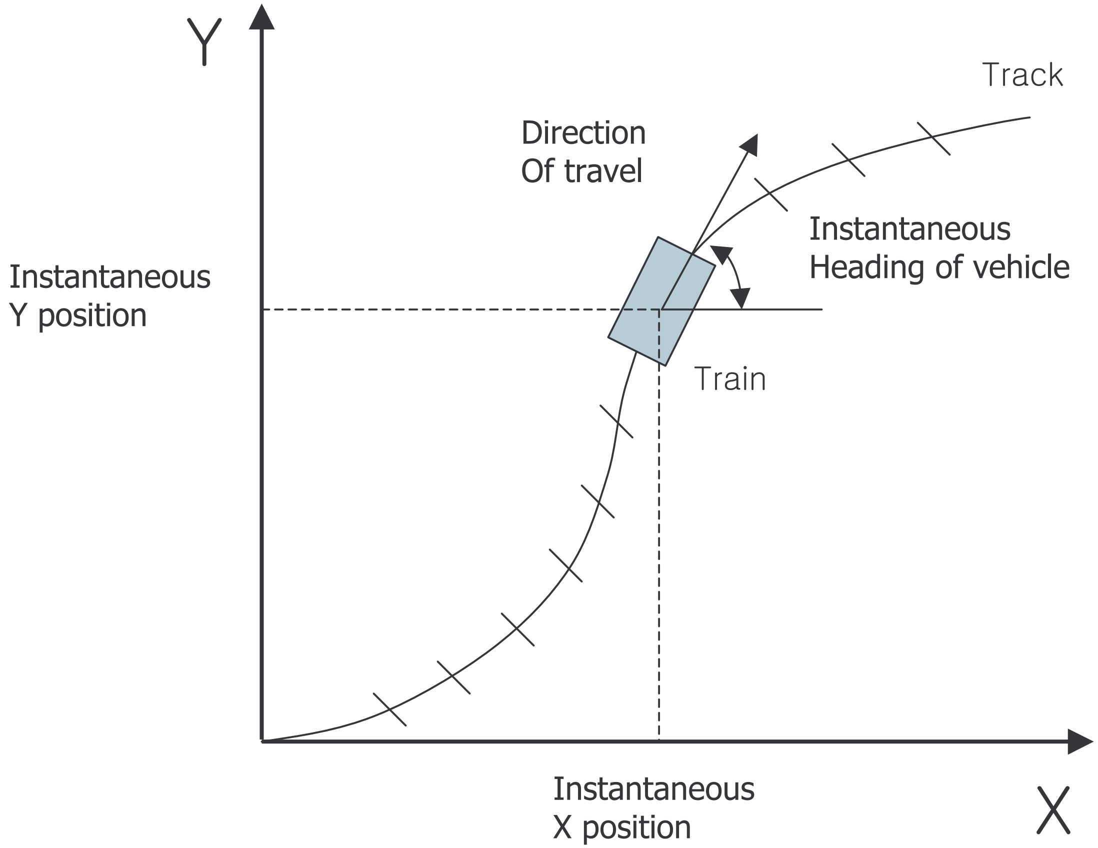
- **Geometry**



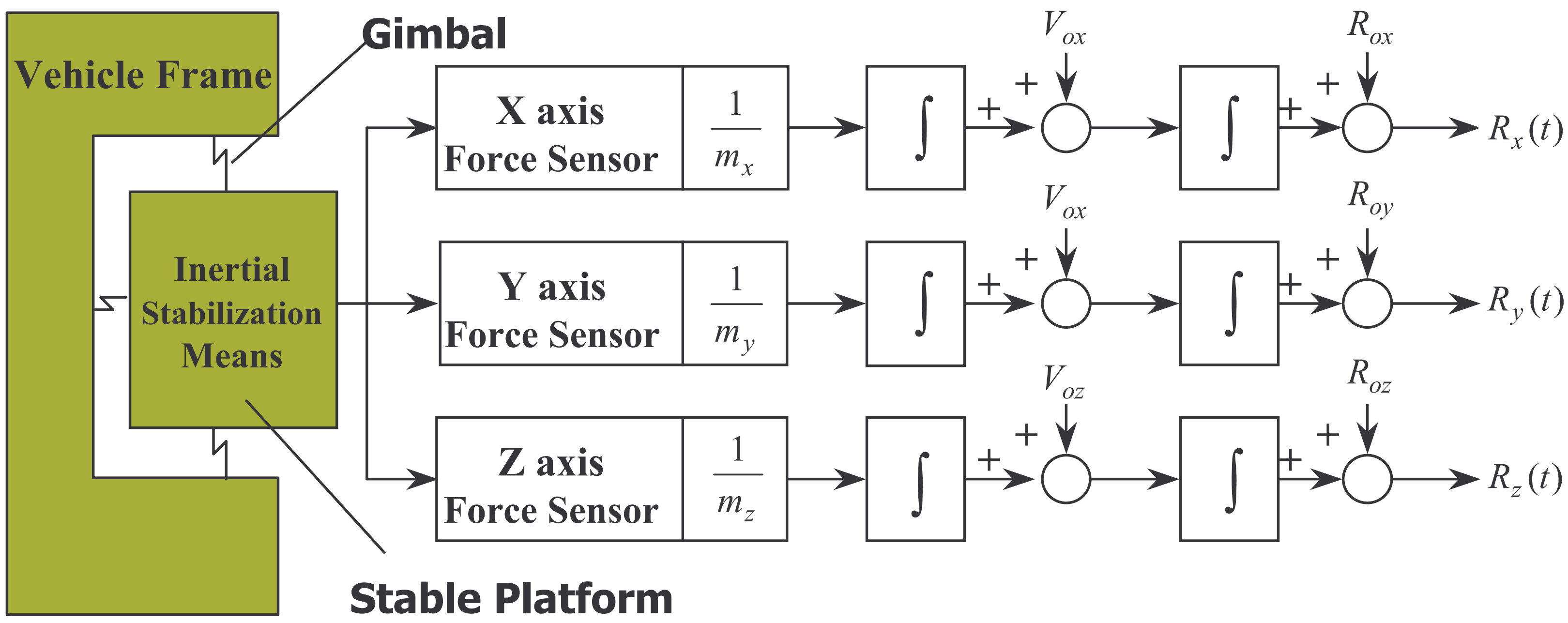
- **Mechanization**



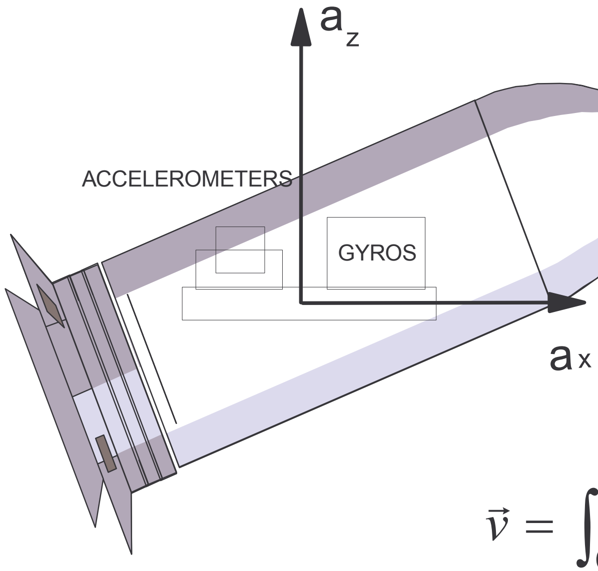
2-D Navigation

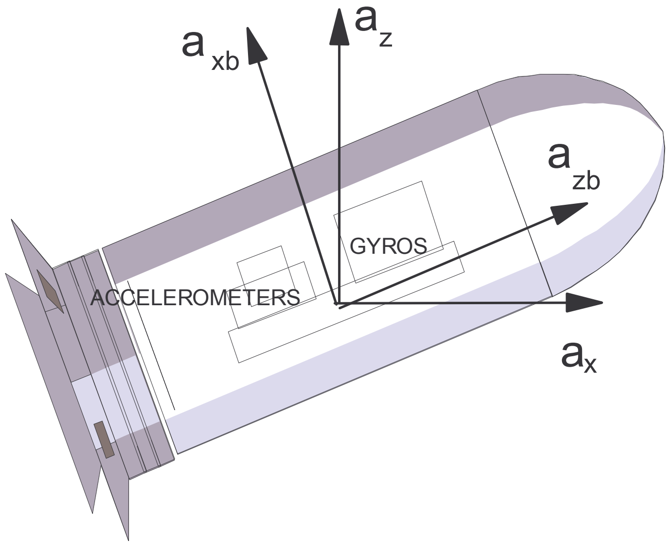


Simple Mechanism of INS



Gimballed INS vs. Strapdown INS

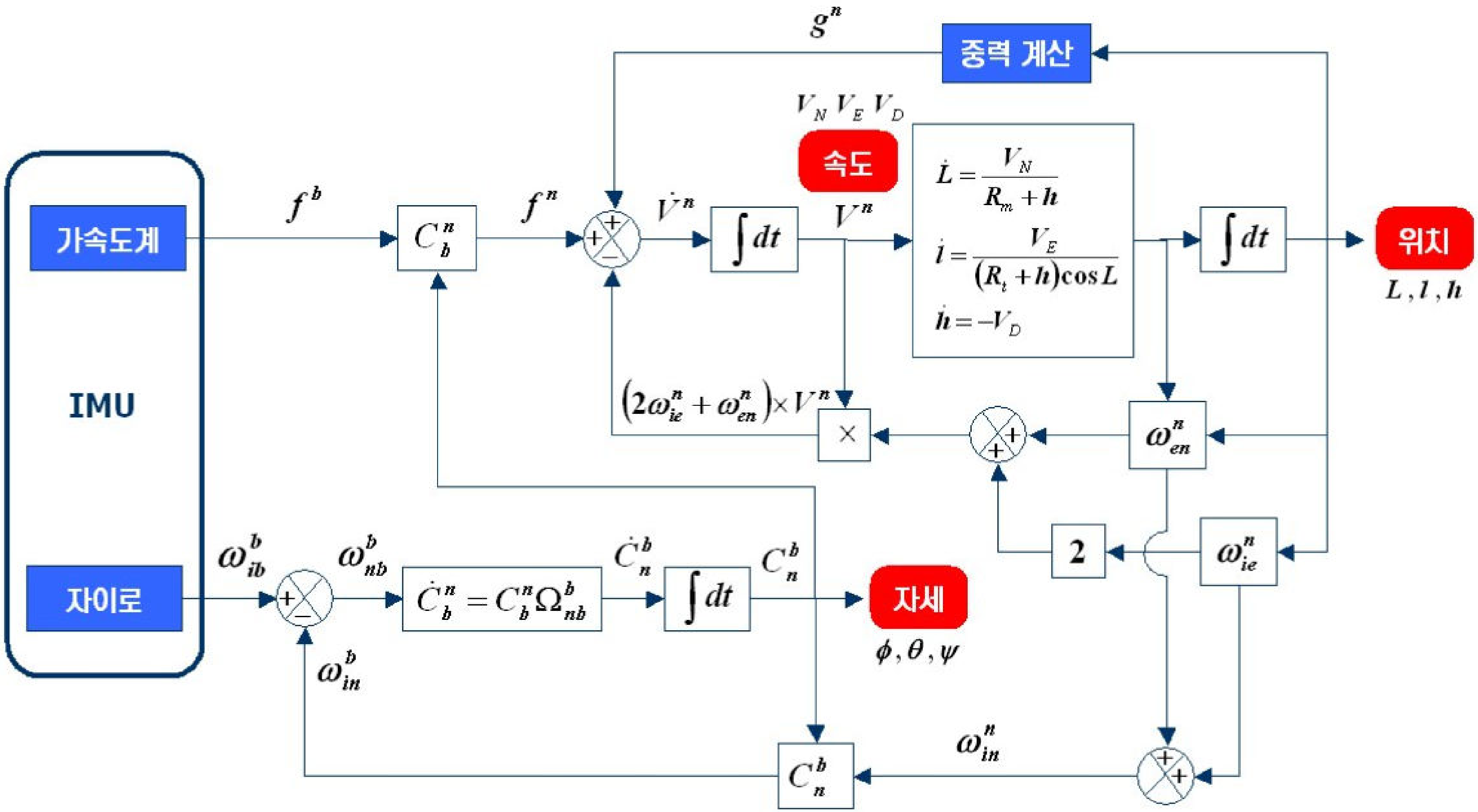




$$\vec{v} = \int_0^t \vec{a}\,dt + \vec{v}_0$$

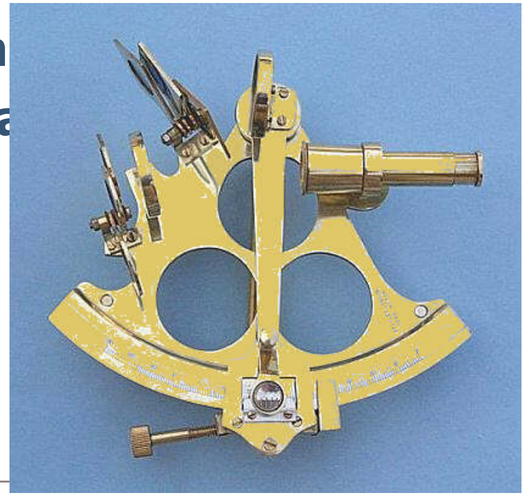
$$\vec{R} = \int_0^t \vec{v}\,dt + \vec{R}_0$$

Strapdown INS(SDINS)



Historical Development(1)

- Earliest times
  - **Navigation by observation**
  - **Polynesians cross the Pacific Ocean about two millennia ago**
- 13th century
  - **Compass, which could be used irrespective of visibility**
  - **Sextant, which enabled position fixes to be made accurately on land**
- 17th century
  - **Isaac Newton defined the laws of mechan and gravitation, which are the fundamenta principles of inertial navigation**
- Early 18th century
  - **A stabilized sextant by Serson**
  - **An accurate chronometer by Harrison**



Historical Development(2)

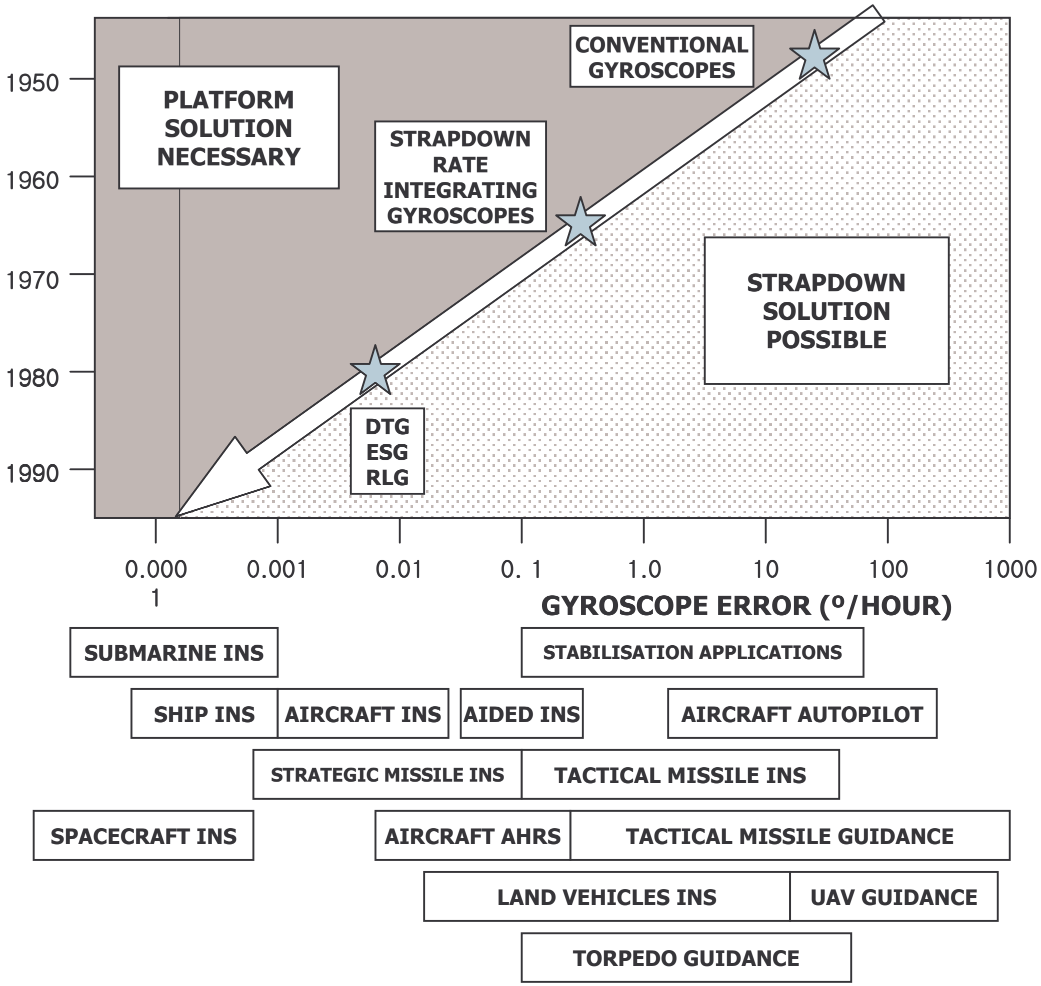
- 19th century
  - **Gyroscopic effect in 1852 by Foucault**
  - **Rotational motion of the Earth and the demonstration of rotational dynamics by Bohneberger, Johnson, and Lemarle**
  - **Ringing of hollow cylinders, a phenomenon later applied to solid state gyroscopes, in 1890 by Prof. G.H.Bryan**
- 20th century
  - **Gyrocompass**
  - **Schuler tuning:** $2\pi\sqrt{R/g} \cong$ **84min, undamped natural period**
  - **Application of the gyroscopic effect to control and guidance by Sperry brother**
  - **Stable platform for fire control systems for guns on ship in the 1920s**
  - **Demonstration of the principles of inertial guidance in the V2 rocket during World War II**
  - **Strapdown technique for navigation in 1949**

항법 및 정보시스템 연구실
Navigation & Information Systems Laboratory
http :// eea.hangkong.ac.kr / nisl /

Historical Development(3)

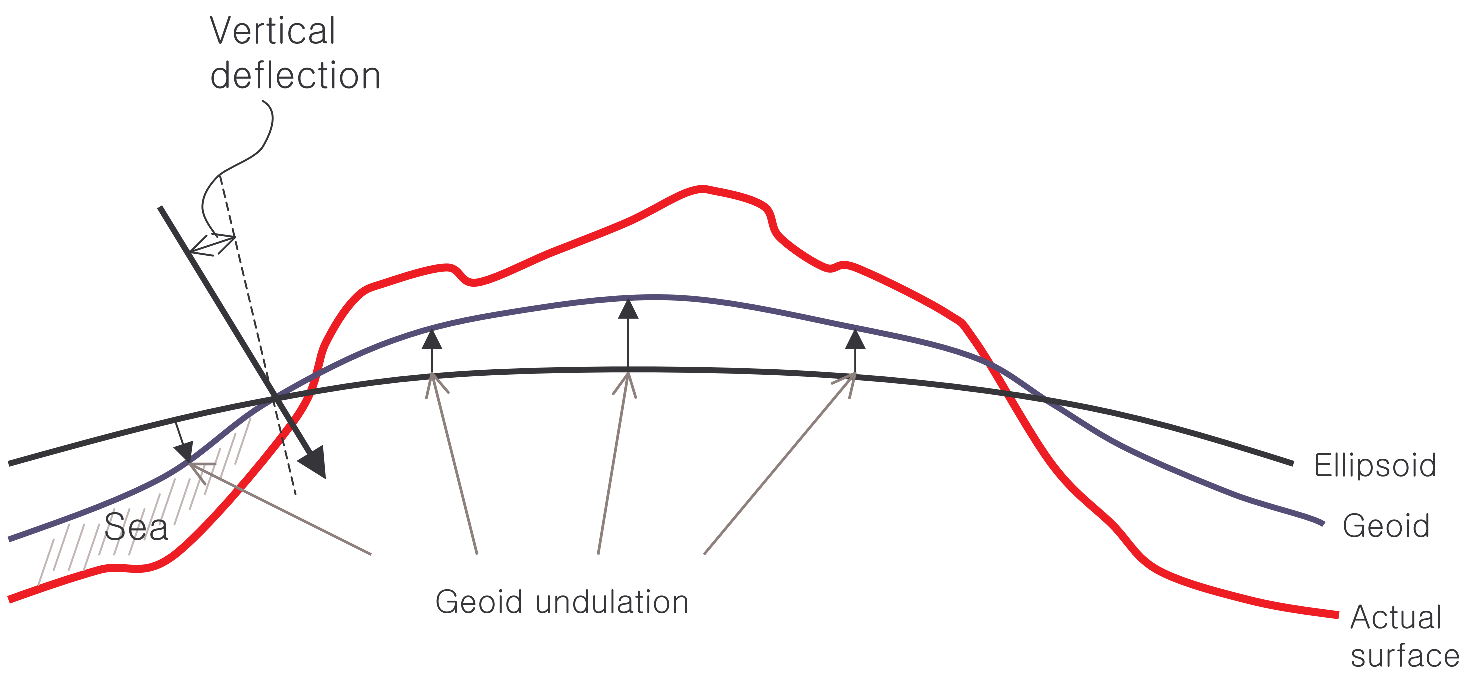
- 20th century(cont'd)
  - **More accurate sensors in the 1950s : 15deg/hr → 0.01deg/hr by Prof. Charles Stark Draper, MIT**
  - **INS using the so-called stable platform technology became standard equipment in military aircraft, ships and submarines during the 1960s**
  - **Ring laser gyroscope**
  - **Ballistic missile and exploration of space**
  - **In the last two decades, the application of the microcomputer**
  - **Development of gyroscopes with large dynamic ranges enabling the strapdown principle to be realized**
- Modern-day inertial Navigation system
  - **Diverse applications : robotics, racing or high performance motor car and for surveying underground well and pipelines**
  - **Call for navigation systems having a very broad range of performance capabilities**

Strapdown Sensor and Applications



Earth

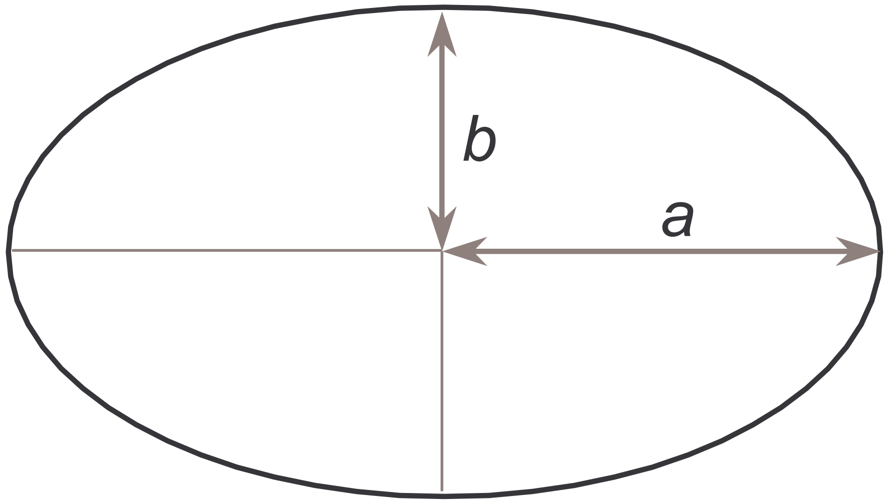
Earth surface, Geoid, and Ellipsoid



- Geoid is defined as level surface of gravity field with best fit to mean sea level. (Maximum difference between geoid and mean sea level is about 1 m)

- Ellipsoid defines an approximated surface to simplify geometries and computations regarding the Earth.

WGS-84 Ellipsoid (Earth Model)

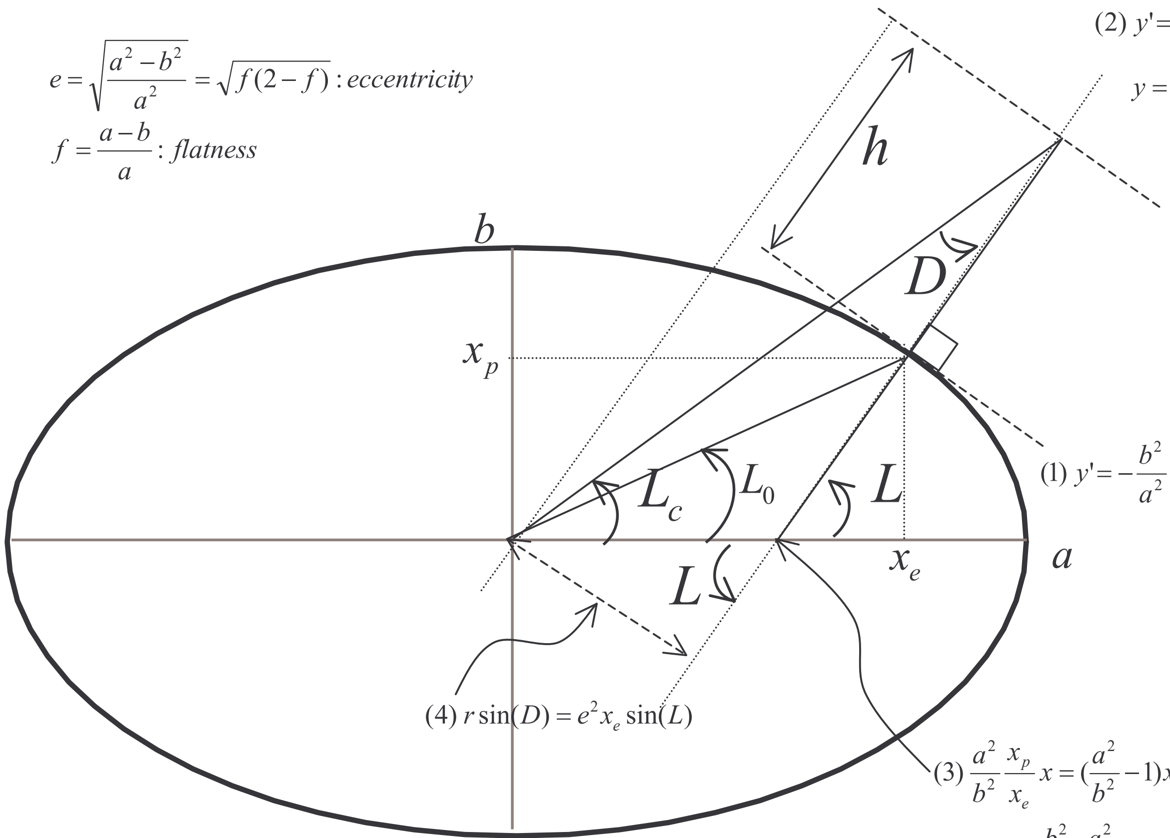


semi-major Axis: $a = 6378137$ (m)
semi-minor Axis: $b = 6356752.3142$ (m)
flatness: $f = (a-b)/a = 1/298.257223563 = 0.00335281066475$
eccentricity: $e=[\ f(2-f)\ ]^{1/2} = 0.08181919084262$

Geodetic and Geocentric Latitudes

$$e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{f(2-f)} : eccentricity$$

$$f = \frac{a-b}{a} : flatness$$



$$(1)\ y' = -\frac{b^2}{a^2}\frac{x_e}{x_p}$$

$$(2)\ y' = \frac{a^2}{b^2}\frac{x_p}{x_e},$$

$$y = \frac{a^2}{b^2}\frac{x_p}{x_e}x + (1 - \frac{a^2}{b^2})x_p$$

$$(3)\ \frac{a^2}{b^2}\frac{x_p}{x_e}x = (\frac{a^2}{b^2} - 1)x_p \Rightarrow \frac{a^2}{b^2}\frac{1}{x_e}x = (\frac{a^2}{b^2} - 1)$$

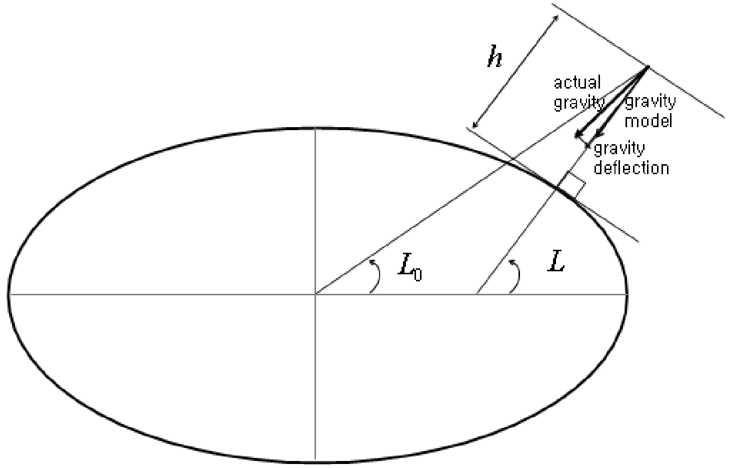
$$\Rightarrow x = \frac{b^2}{a^2}(\frac{a^2}{b^2} - 1)x_e \Rightarrow x = e^2 x_e$$

$$(4)\ r\sin(D) = e^2 x_e \sin(L)$$

$$(5)\ D = \sin^{-1}\left[\frac{e^2 x_e \sin(L)}{r}\right] : latitude\ difference$$

Gravity Model

$$g^n = \begin{bmatrix} 0 \\ 0 \\ g_z(L,h) \end{bmatrix} + g_{anomaly}$$



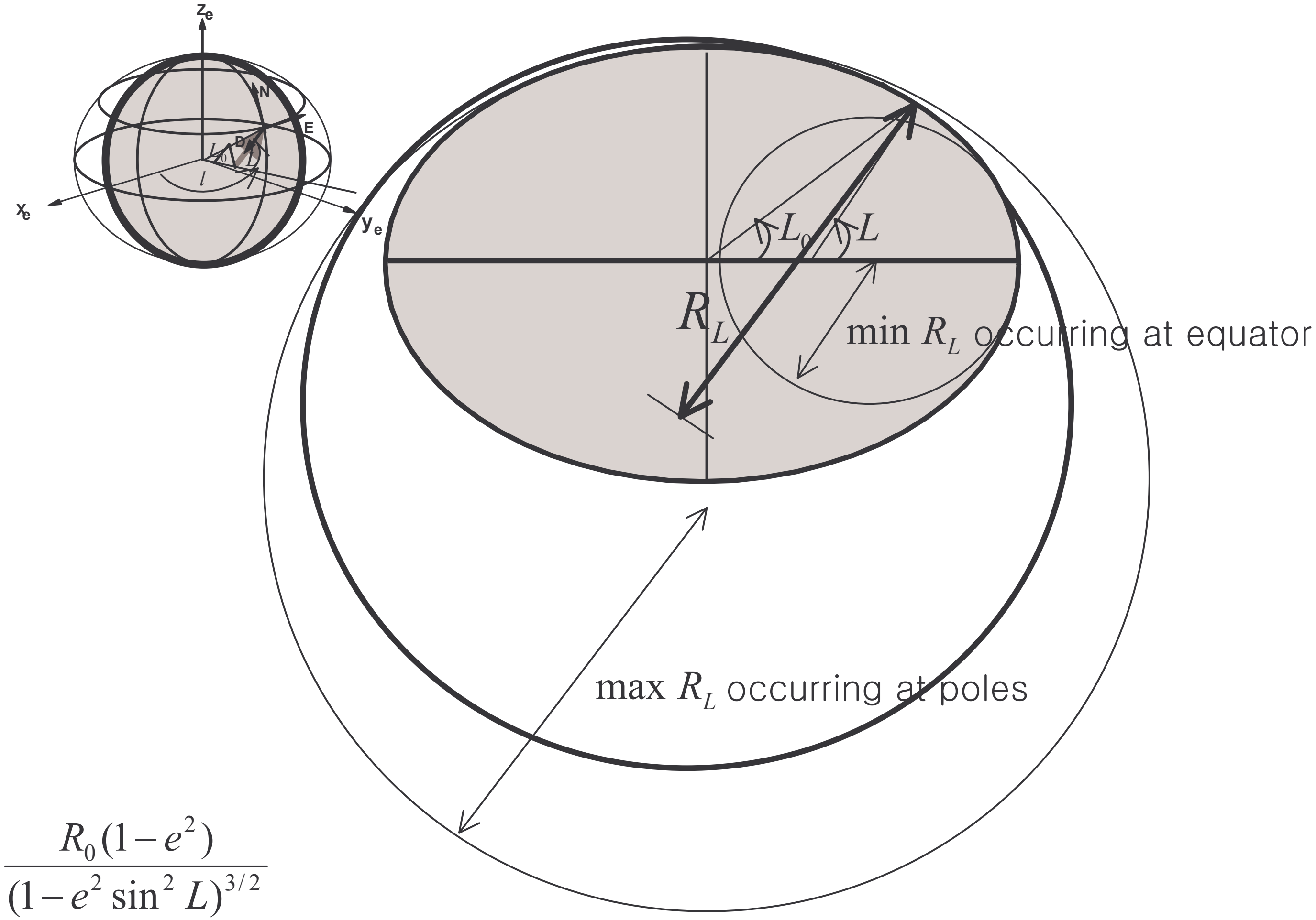
$$g_z(L,h) = g_z(L) - [3.0877\times10^{-6} - 0.0044\times10^{-6}\sin^2(L)]h + 0.072\times10^{-12}h^2 \quad (m/s^2)$$

$$g_z(L) = g_0[1 + 0.0053024\sin^2(L) - 0.0000058\sin^2(2L)] \quad (m/s^2)$$

$$g_0 = 9.780327 \quad (m/s^2)$$

* h is measured in m .
* Magnitude of $g_{anomaly} := [\varsigma_{anomaly} \quad -\eta_{anomaly} \quad \delta_{anomaly}]^T$ caused by the gravity deflection is typically less than $10^{-5} g_0$.

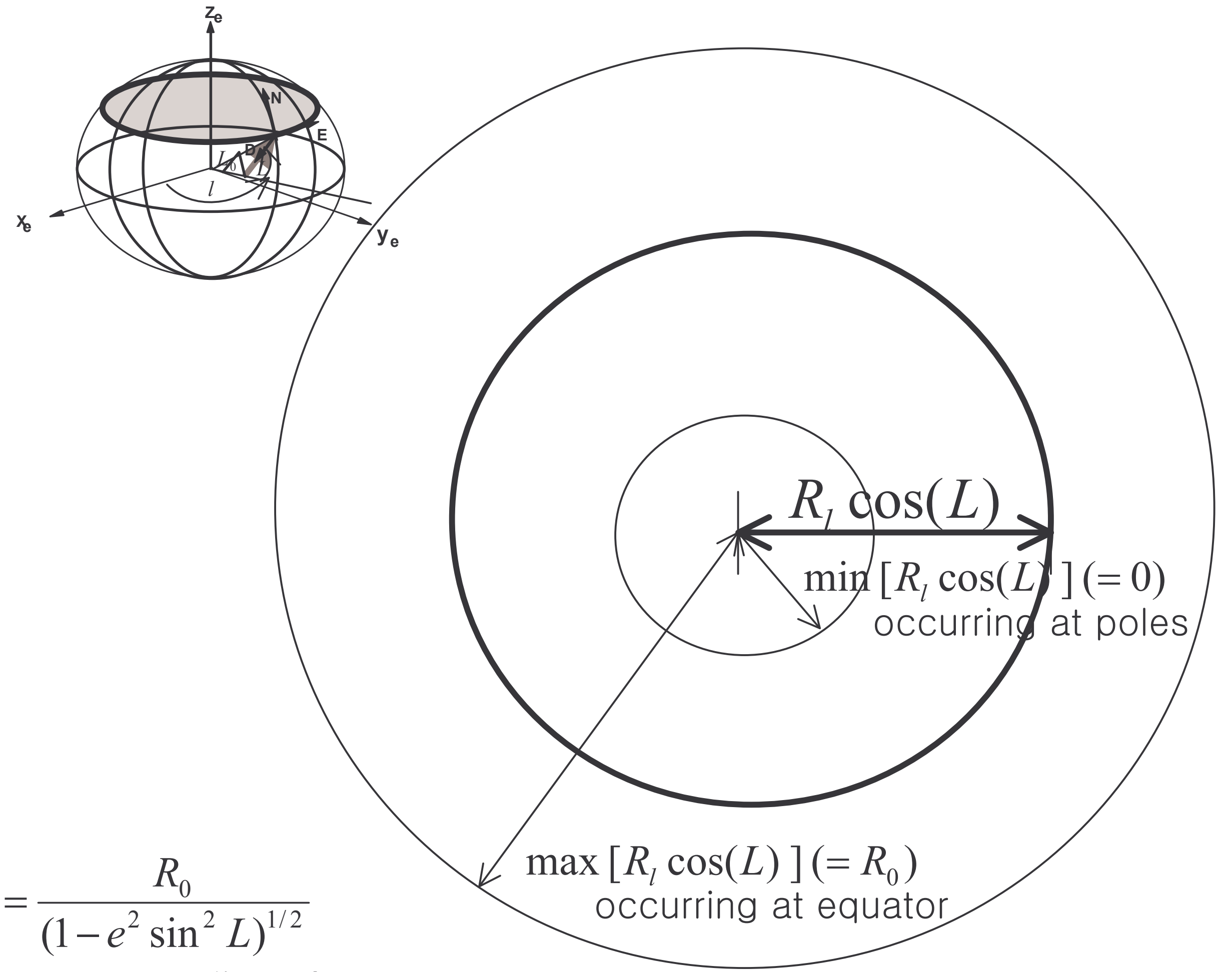
Meridian Radius of Curvature



$$R_L = \frac{R_0(1-e^2)}{(1-e^2 \sin^2 L)^{3/2}}$$

: meridian radius of curvature at a given latitude

Transverse Radius of Curvature



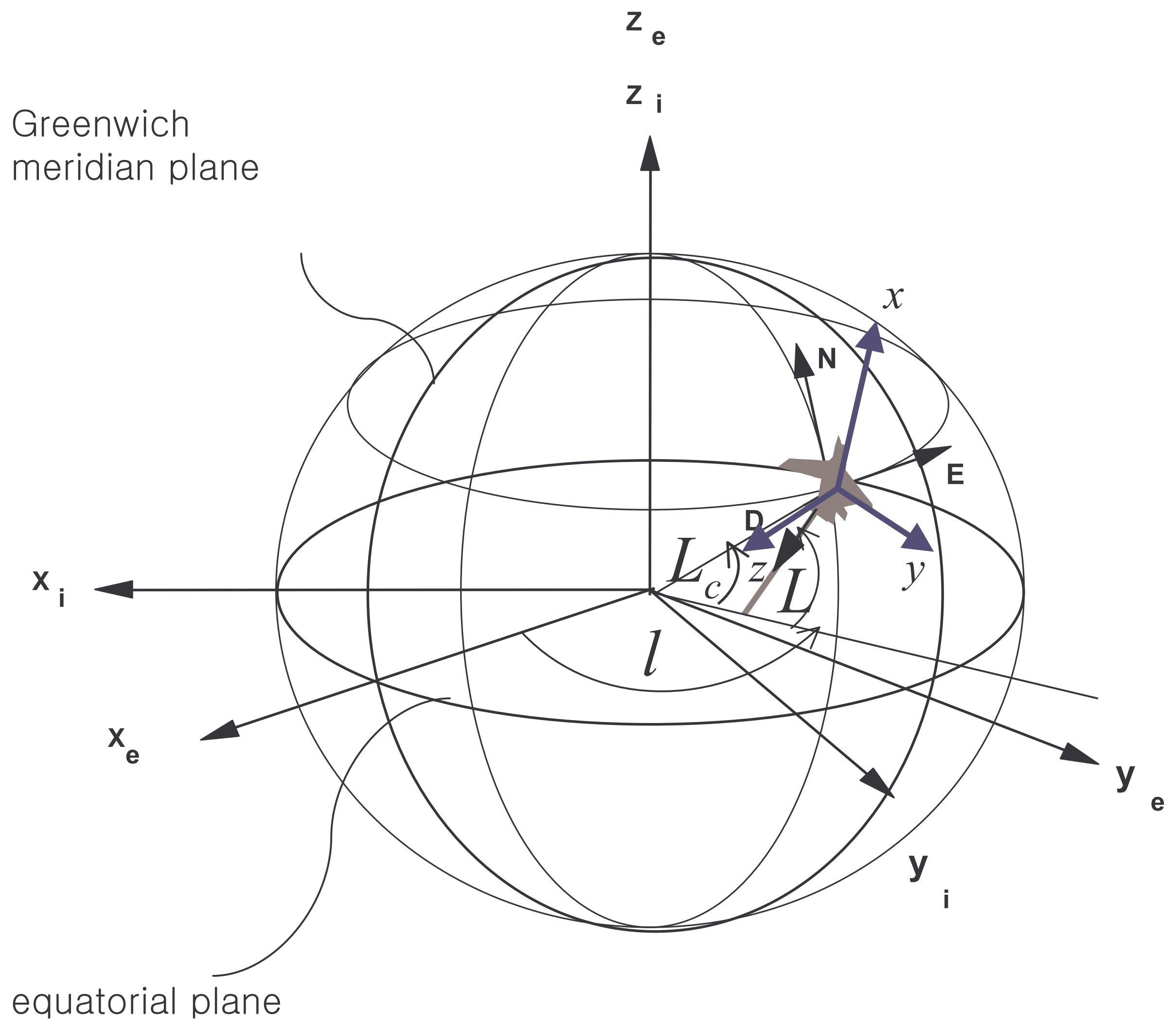
$$R_l = \frac{R_0}{(1-e^2 \sin^2 L)^{1/2}}$$

: transverse radius of curvature

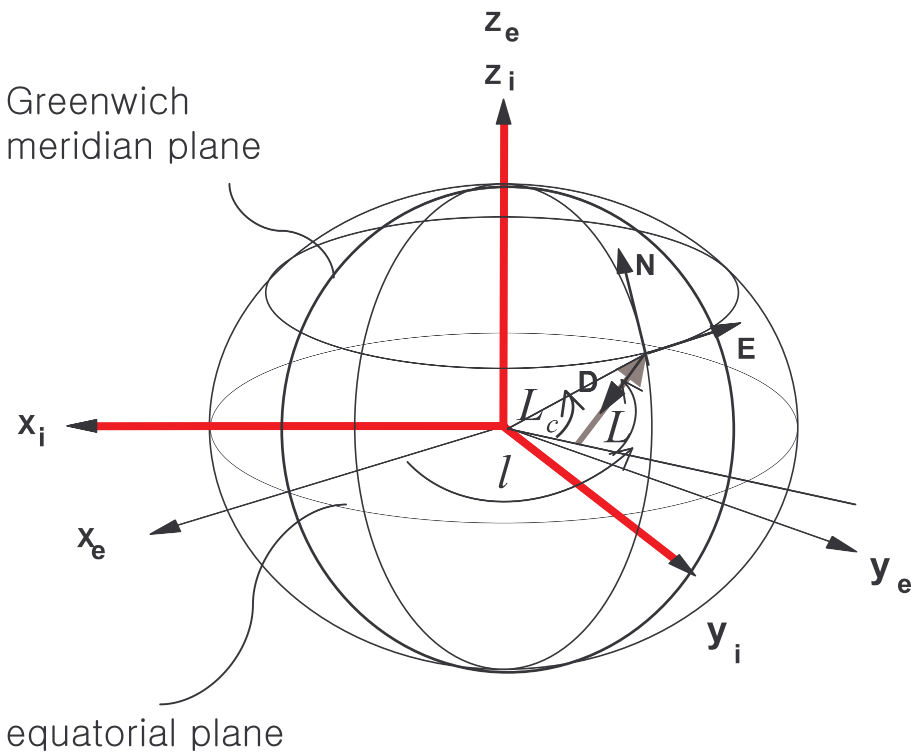
Coordinate Systems

항법 및 정보시스템 연구실
Navigation & Information Systems Laboratory
항공전자 및 정보통신공학부
한국항공대학교
HANKUK AVIATION UNIVERSITY

Coordinate systems related to INS

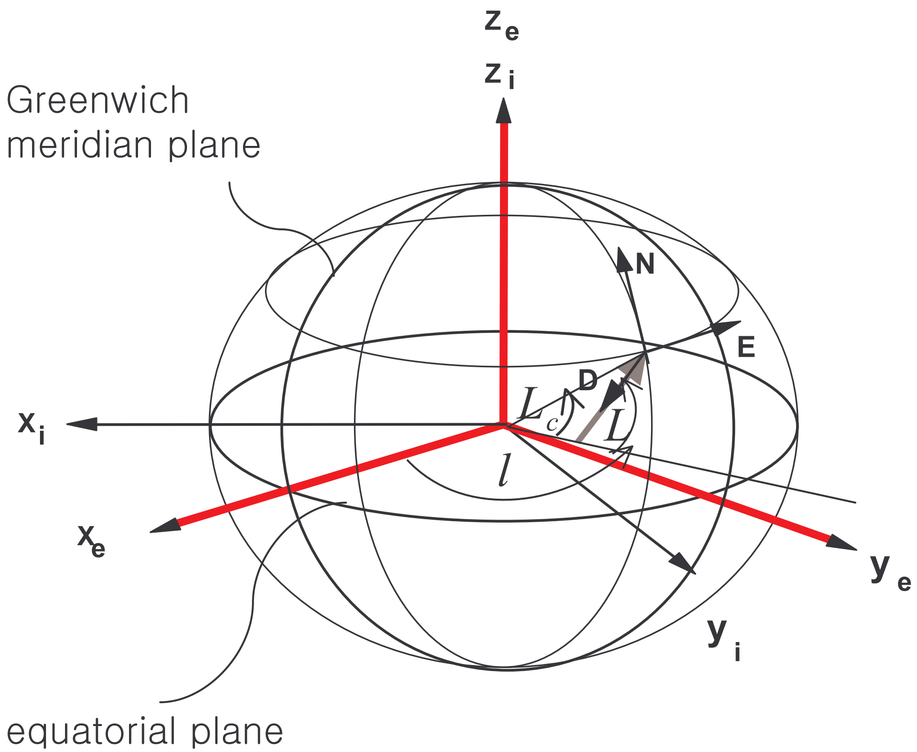


Coordinate Systems: Inertial Frame (i-frame)



- Inertial Frame (i-frame) – A reference frame in which Newton's laws of motion apply. Non-accelerating but may be in uniform linear motion. An orthogonal coordinate system.

Coordinate Systems: ECEF Frame (e-frame)

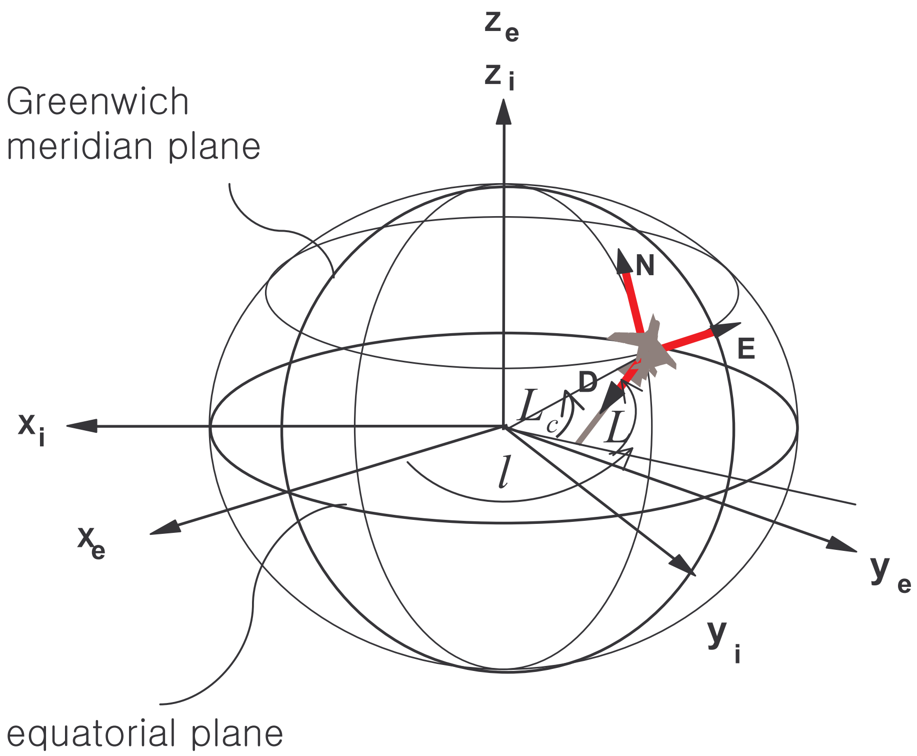


- Earth-Centered Earth-Fixed (ECEF) Frames (e-frame) – Its origin fixed to the center of the earth. The axes rotate relative to the inertial frame with a frequency of

$$\omega_{ie} \approx \frac{1+365.25\ cycles}{(365.25)(24)hr}\frac{2\pi\ rad/cycle}{3600\ \sec/hr} \approx 7.292115\times10^{-5}\ rad/\sec$$

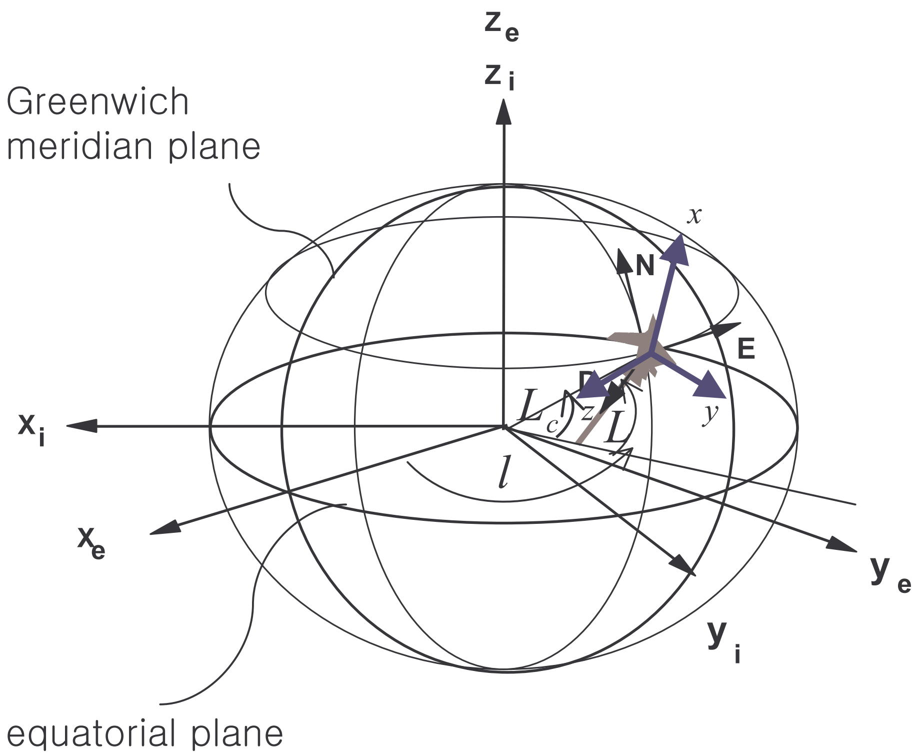
- because of the daily earth rotation and yearly revolution about the sun.

Coordinate Systems: Locally-Level Frame (n-frame)



- Locally-Level Frame (n-frame) – The z-axis points toward the interior of the ellipsoid along the ellipsoid normal. The x-axis points toward true north. The y-axis follows the right-handed rule.

Coordinate Systems: Body Frame (b-frame)



- Body Frame (b-frame) – The origin is usually at the center of gravity of the vehicle of interest. The x-axis is defined in the forward direction. The z-axis is defined pointing to the bottom of the vehicle. The y-axis completes the right-handed orthogonal coordinate system.

Coordinate Transformation

항법 및 정보시스템 연구실
Navigation & Information Systems Laboratory
항공전자 및 정보통신공학부
한국항공대학교
HANKUK AVIATION UNIVERSITY

Transformation about a Single-Axis

- A non-zero vector fixed in space can be expressed with respect to various frames.
- If we express with respect to the i-frame and e-frame, they are summarized as

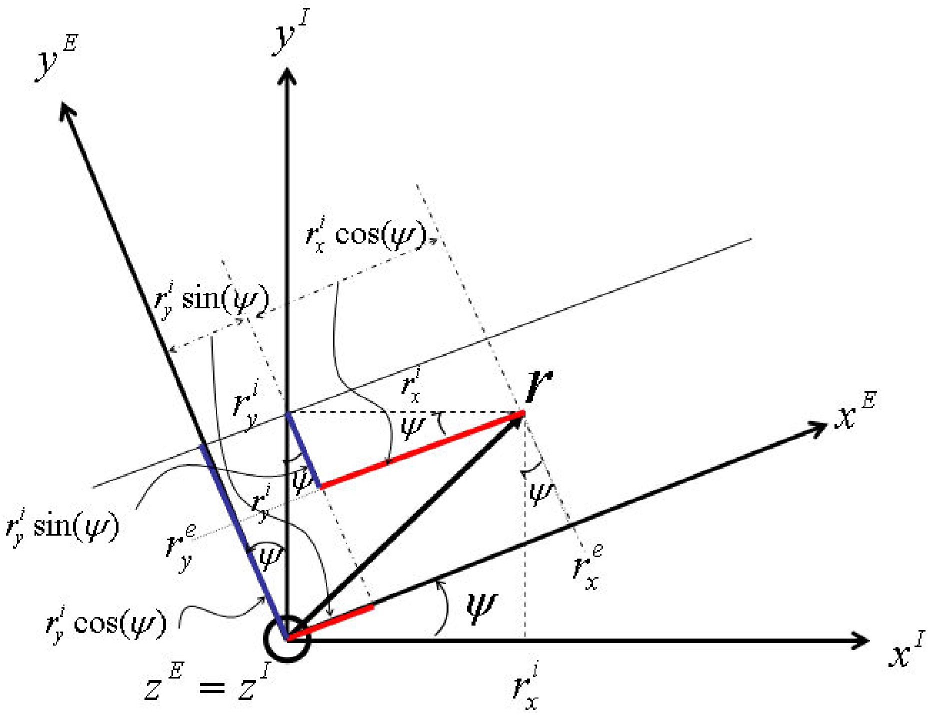
$$r^i = \begin{bmatrix} r_x^i \\ r_y^i \\ r_z^i \end{bmatrix}, \qquad r^e = \begin{bmatrix} r_x^e \\ r_y^e \\ r_z^e \end{bmatrix}$$

$$r^e = C_i^e r^i, \quad C_i^e = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$r_x^e = r_x^i \cos\psi + r_y^i \sin\psi$$

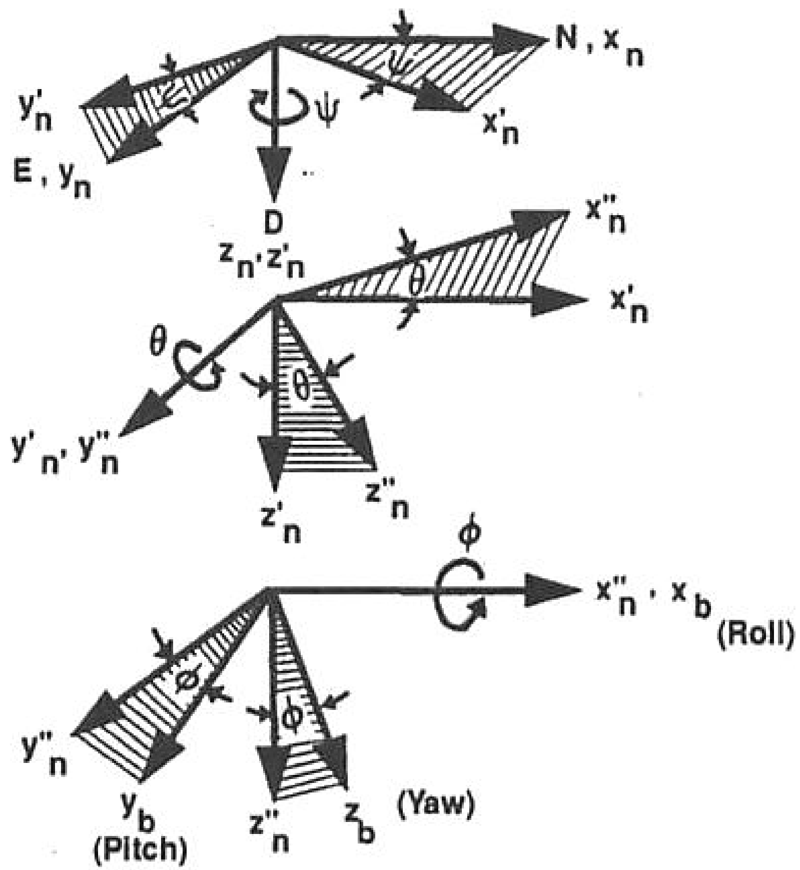
$$r_y^e = -r_x^i \sin\psi + r_y^i \cos\psi$$

$$r_z^e = r_z^i$$



Successive Rotations: Euler Angles

- Euler angles from n-frame to b-frame:
  (1) ψ- rotation about z
  (2) θ- rotation about y'
  (3) ϕ - rotation about x″



Thus, the total transformation matrix can be decomposed by three elementary transformation matrices as follows.

$$C_n^b = C_{n''}^b C_{n'}^{n''} C_n^{n'}$$

where

$$C_n^{n'}(\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C_{n'}^{n''}(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad C_{n''}^b(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

Therefore,

$$C_n^b = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix}$$

Properties of the Transformation Matrix

(1) Inverse transformation

$$C_m^i = (C_i^m)^{-1}$$

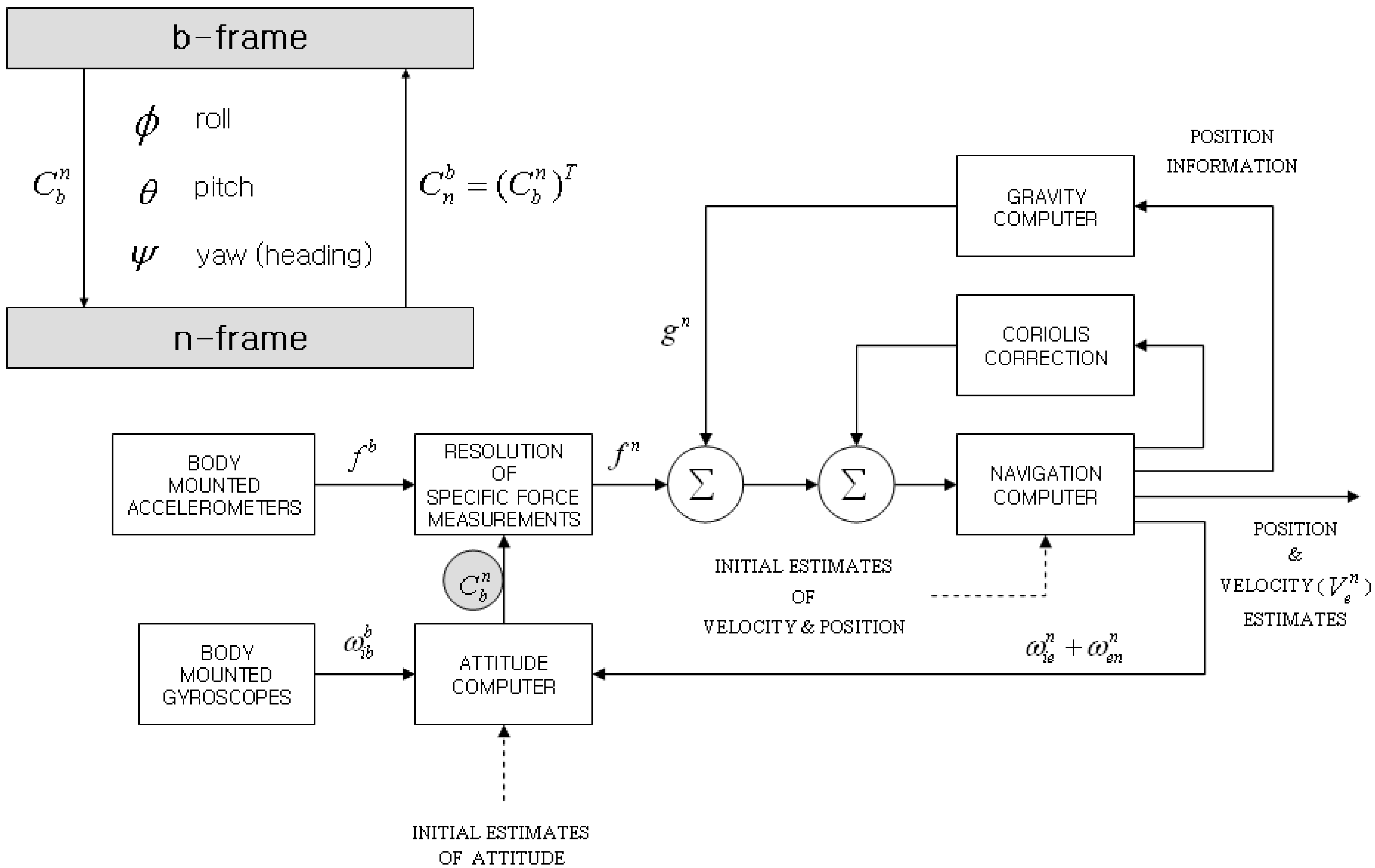
(2) Orthonormality

$$(C_i^m)^T C_i^m = I$$

(3) Transpose matrix equals inverse matrix by (1) and (2)

$$(C_i^m)^T = (C_i^m)^{-1} = C_m^i$$

Utilization of Transformation Matrix in SDINS



Attitude Differential Equation

항법 및 정보시스템 연구실
Navigation & Information Systems Laboratory
항공전자 및 정보통신공학부
한국항공대학교
HANKUK AVIATION UNIVERSITY

Attitude Differential Equation

$$\dot{C}_b^n = C_b^n \left\langle \omega_{nb}^b \right\rangle$$

From the definition of differential equations, it is obvious that

$$\dot{C}_b^n = \lim_{\Delta t \to 0} \frac{\Delta C_b^n}{\Delta t} = \lim_{\Delta t \to 0} \frac{C_b^n(t+\Delta t) - C_b^n(t)}{\Delta t}. \qquad (1)$$

$C_b^n(t+\Delta t)$ can be decomposed as follows where small angle approximation is utilized (i.e., products of small angles are neglected).

$$C_b^n(t+\Delta t) = C_b^n(t)C(\delta\psi)C(\delta\theta)C(\delta\phi) = C_b^n(t)\,[I_{3\times 3} + \delta C] \qquad (2)$$

where

$$C(\psi) = \begin{bmatrix} \cos\delta\psi & \sin\delta\psi & 0 \\ -\sin\delta\psi & \cos\delta\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cong \begin{bmatrix} 1 & \delta\psi & 0 \\ -\delta\psi & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C(\phi) \cong \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \delta\phi \\ 0 & -\delta\phi & 1 \end{bmatrix},$$

$$C(\theta) \cong \begin{bmatrix} 1 & 0 & -\delta\theta \\ 0 & 1 & 0 \\ \delta\theta & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} \delta\phi \\ \delta\theta \\ \delta\psi \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \Delta t = \omega_{nb}^b \,\Delta t \qquad (3)$$

By combining Eqs. (2) and (3), it is easy to verify that

$$C(\delta\psi)C(\delta\theta)C(\delta\phi) = I_{3\times3} + \delta C\,, \qquad (4)$$

where

$$\delta C = \begin{bmatrix} 0 & -\delta\psi & \delta\theta \\ \delta\psi & 0 & -\delta\phi \\ -\delta\theta & \delta\phi & 0 \end{bmatrix} = \left\langle \omega_{nb}^{b} \right\rangle \Delta t\,, \qquad (5)$$

$\langle\omega\rangle$: 3×3 skew-symmetric matrix based on 3×1 vector angle ω.

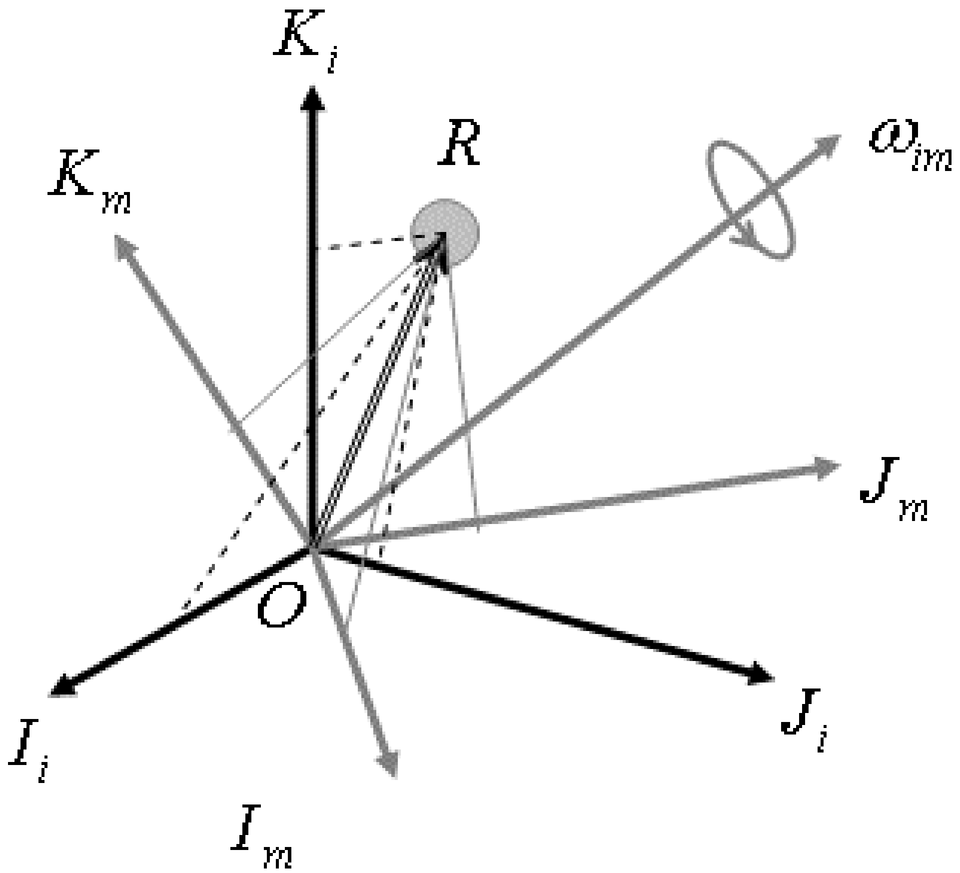
Substituting Eqs. (2) and (5) to (1) yields

$$\dot{C}_b^n = \lim_{\Delta t \to 0} \frac{C_b^n \left\langle \omega_{nb}^{b} \right\rangle \Delta t}{\Delta t} = C_b^n \left\langle \omega_{nb}^{b} \right\rangle \qquad (6)$$

Position and Velocity Differential Equations

항법 및 정보시스템 연구실
Navigation & Information Systems Laboratory
항공전자 및 정보통신공학부
한국항공대학교
HANKUK AVIATION UNIVERSITY

Parameterization, Differentiation, and Frames



- **Vector**
  - **an arrow (rod) consisting of starting and finishing points**

$$R$$

- **Parameterization of a vector w.r.t. a frame**
  - **The same vector can be represented by different parameterizations if reference frames are different.**

$$R^m = \begin{bmatrix} x_m \\ y_m \\ z_m \end{bmatrix} = x_m I_m + y_m J_m + z_m K_m$$

- **Differentiation of a vector w.r.t. a frame**
  - **Differentiation of the same vector can result in different vectors if reference frames for differentiation are different. We ride the reference frame for differentiation and watch the changes of the vector.**

$$\left[\frac{dR}{dt}\right]_m$$

- **Differentiation of a vector w.r.t. a frame and its parameterization w.r.t. another frame**

$$\left(\left[\frac{dR}{dt}\right]_m\right)^j$$

Differentiation w.r.t. Different Frames

Given $R^i = C_m^i R^m$ **(7)**

differentiate $\dot{R}^i = \dot{C}_m^i R^m + C_m^i \dot{R}^m$ **(8)**

where

$$\dot{R}^m = \begin{bmatrix} \dot{x}_m \\ \dot{y}_m \\ \dot{z}_m \end{bmatrix} = \left(\left[\frac{dR}{dt} \right]_m \right)^m$$

$$\dot{R}^i = \left(\left[\frac{dR}{dt} \right]_i \right)^i$$

$$\dot{C}_m^i = C_m^i \left\langle \omega_{im}^m \right\rangle = C_m^i (\omega_{im}^m \times)$$

(9)

항법 및 정보시스템 연구실
Navigation & Information Systems Laboratory
http :// eea.hangkong.ac.kr / nisl /

Apply (9) to (8):

$$\left(\left[\frac{dR}{dt}\right]_i\right)^i = C_m^i \langle \omega_{im}^m \rangle R^m + C_m^i \left(\left[\frac{dR}{dt}\right]_m\right)^m$$

$$= \langle \omega_{im}^i \rangle C_m^i R^m + C_m^i \left(\left[\frac{dR}{dt}\right]_m\right)^m$$

$$= \left(\langle \omega_{im} \rangle R + \left[\frac{dR}{dt}\right]_m\right)^i \qquad (10)$$

Restore vector parameterization (i-frame) to vector:

$$\left[\frac{dR}{dt}\right]_i = \left[\frac{dR}{dt}\right]_m + \langle \omega_{im} \rangle R \qquad (11)$$

Coriolis Equation

Specific Force Equations in a Moving Frame

For brevity, define the differentiation operators

$$p_i = \left[\frac{d}{dt}\right]_i \qquad p_e = \left[\frac{d}{dt}\right]_e \qquad p_m = \left[\frac{d}{dt}\right]_m \qquad (12)$$

We already know that

$$p_i R^m = p_m R^m + \omega_{im}^m \times R^m \quad \text{(Coriolis equation)} \qquad (13)$$

$$f^m = p_i^2 R^m - G(R)^m \quad \text{(Specific force equation)} \qquad (14)$$

The two equations are combined as follows

$$f^m = p_m p_i R^m + \omega_{im}^m \times p_i R^m - G(R)^m$$

$$\left\langle * \; p_i R^m = p_e R^m + \omega_{ie}^m \times R^m \right\rangle$$

$$= p_m p_e R^m + p_m(\omega_{ie}^m \times R^m) + \omega_{im}^m \times (p_e R^m + \omega_{ie}^m \times R^m) - G(R)^m$$

$$= p_m p_e R^m + (p_m \omega_{ie}^m) \times R^m + \omega_{ie}^m \times (p_m R^m) + \omega_{im}^m \times (p_e R^m) + \omega_{im}^m \times (\omega_{ie}^m \times R^m) - G(R)^m$$

$$= p_m p_e R^m + (p_m \omega_{ie}^m) \times R^m + \omega_{ie}^m \times (p_e R^m) + \omega_{ie}^m \times (\omega_{me}^m \times R^m)$$
$$+ \omega_{im}^m \times (p_e R^m) + \omega_{im}^m \times (\omega_{ie}^m \times R^m) - G(R)^m$$

$\langle * \; p_i \omega_{ie}^m = O \rangle$

$\langle * \; p_m \omega_{ie}^m = p_i \omega_{ie}^m + \omega_{mi}^m \times \omega_{ie}^m = p_i \omega_{ie}^m - \omega_{im}^m \times \omega_{ie}^m = -\omega_{im}^m \times \omega_{ie}^m \rangle$

$$= p_m p_e R^m - (\omega_{im}^m \times \omega_{ie}^m) \times R^m + \omega_{ie}^m \times (p_e R^m) + \omega_{ie}^m \times (\omega_{me}^m \times R^m)$$
$$+ \omega_{im}^m \times (p_e R^m) + \omega_{im}^m \times (\omega_{ie}^m \times R^m) - G(R)^m$$

$\langle * \; \omega_{im}^m \times (\omega_{ie}^m \times R^m) = \omega_{ie}^m \times (\omega_{im}^m \times R^m) + (\omega_{im}^m \times \omega_{ie}^m) \times R^m \rangle$

$$= p_m p_e R^m + \omega_{ie}^m \times (p_e R^m) + \omega_{ie}^m \times (\omega_{me}^m \times R^m) + \omega_{im}^m \times (p_e R^m) - G(R)^m$$
$$+ \omega_{ie}^m \times (\omega_{im}^m \times R^m)$$

$\langle * \; \omega_{im}^m + \omega_{me}^m = \omega_{ie}^m \rangle$

$$= p_m p_e R^m + \omega_{ie}^m \times (p_e R^m) + \omega_{ie}^m \times (\omega_{ie}^m \times R^m) + \omega_{im}^m \times (p_e R^m) - G(R)^m$$

$$= p_m p_e R^m + [(\omega_{ie}^m \times) + (\omega_{im}^m \times)] p_e R^m - g^m$$

$$= p_m p_e R^m + [2(\omega_{ie}^m \times) + (\omega_{em}^m \times)] p_e R^m - g^m \quad \textbf{(15)}$$

where the gravity is defined from the gravitational acceleration as follows.

$$g^m = G(R)^m - \omega_{ie}^m \times \omega_{ie}^m \times R^m \quad \textbf{(16)}$$

Velocity Differential Equation

$$\dot{V}^n = C_b^n f^b - [2(\omega_{ie}^n \times) + (\omega_{en}^n \times)]V^n + g^n \quad (17)$$

Set $m = n$ and define

$$V^n = [V_N \quad V_E \quad V_D]^T := p_e R^n = \left(\left[\frac{dR}{dt}\right]_e\right)^n \quad (18)$$

Then

$$p_n p_e R^n = \dot{V}^n \quad (19)$$

$$\dot{V}^n = f^n - [2(\omega_{ie}^n \times) + (\omega_{en}^n \times)]V^n + g^n \quad (20)$$

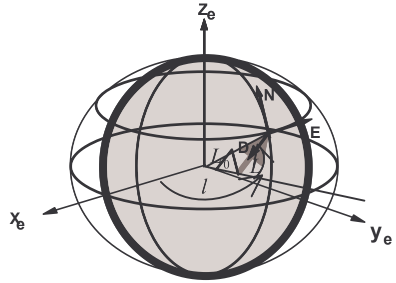
where

$$\omega_{ie}^n = \begin{bmatrix} \Omega_N \\ 0 \\ \Omega_D \end{bmatrix} = \begin{bmatrix} \Omega \cos L \\ 0 \\ -\Omega \sin L \end{bmatrix}, \quad \Omega := \|\omega_{ie}\|, \quad \omega_{en}^n = \begin{bmatrix} \rho_N \\ \rho_E \\ \rho_D \end{bmatrix} = \begin{bmatrix} V_E/(R_l + h) \\ -V_N/(R_L + h) \\ -V_E \tan L/(R_l + h) \end{bmatrix} = \begin{bmatrix} \dot{l} \cos L \\ -\dot{L} \\ -\dot{l} \sin L \end{bmatrix} \quad (21)$$

$$R_L = \frac{R_0(1-e^2)}{(1-e^2 \sin^2 L)^{3/2}}, \quad R_l = \frac{R_0}{(1-e^2 \sin^2 L)^{1/2}} \quad (22)$$

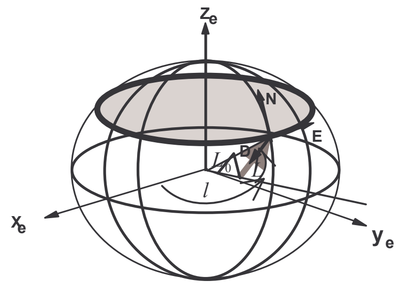
Position Differential Equation

$$\dot{L} = \frac{V_N}{R_L + h}$$



$$R_L = \frac{R_0(1-e^2)}{(1-e^2\sin^2 L)^{3/2}}$$

$$\dot{l} = \frac{V_E}{(R_l + h)\cos L}$$



$$R_l = \frac{R_0}{(1-e^2\sin^2 L)^{1/2}}$$

$$\dot{h} = -V_D$$

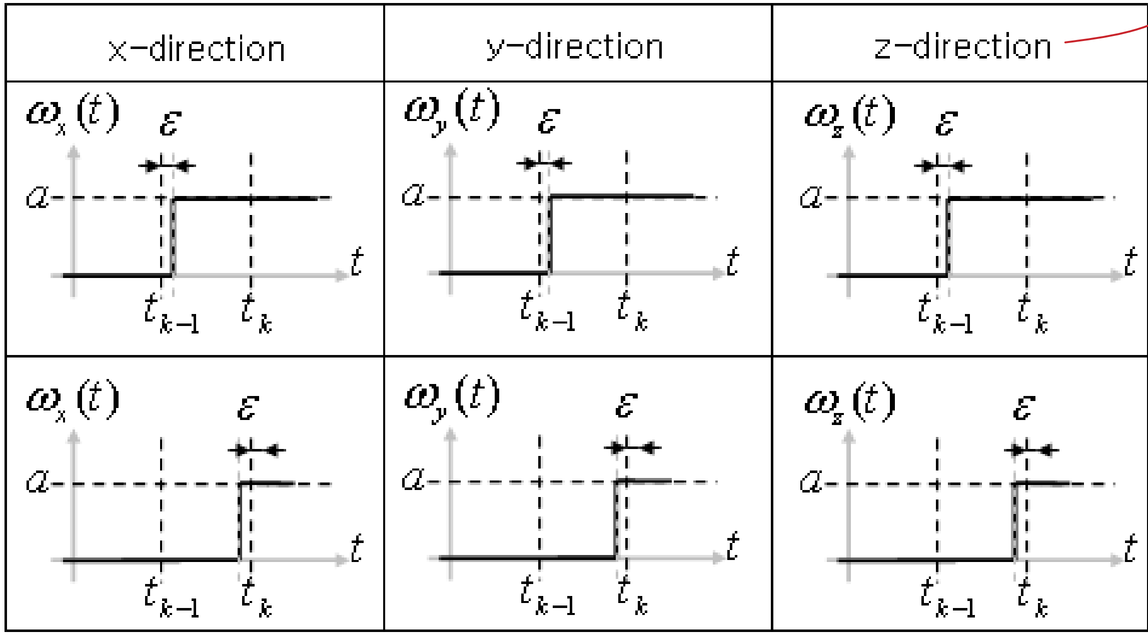
Quaternion-based Attitude Algorithm

항법 및 정보시스템 연구실
Navigation & Information Systems Laboratory
항공전자 및 정보통신공학부
한국항공대학교
HANKUK AVIATION UNIVERSITY

What Is Non-Commutivity Error ?

$$\dot{C}_b^i = C_b^i \langle \omega_{ib}^b \rangle \xrightarrow[\text{for implementation}]{\text{digitization}} C_b^i(t_{k+1}) = C_b^i(t_k)\left(I_{3\times3} + \langle \tilde{\omega}_{ib}^b(t_k) \rangle\right)$$

$$\tilde{\omega}_{ib}^b(t_k) = [\omega_x(t_k) \quad \omega_y(t_k) \quad \omega_z(t_k)]^T = [a \quad a \quad a]^T$$



ambiguous !

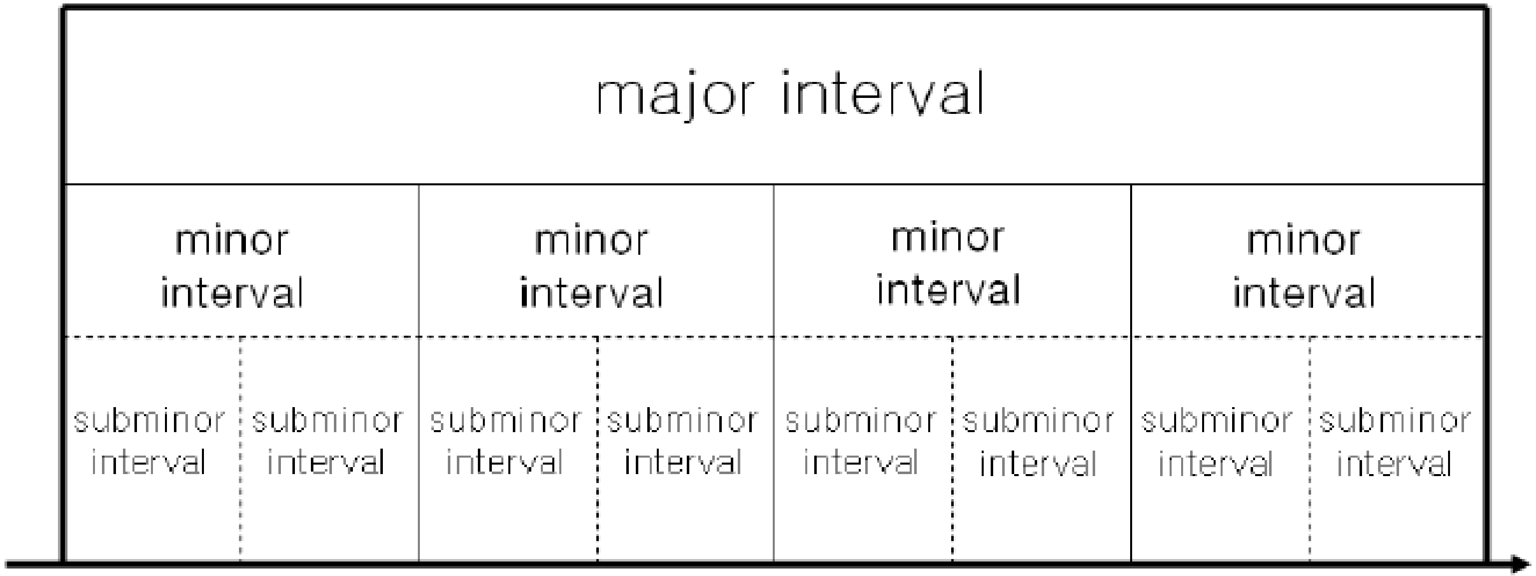
$* \; 0 < \varepsilon < (t_k - t_{k-1})$

Need for Fast Computation

- **Attitude is among most important information provided by inertial navigation systems**
- **The example illustrates how rotation sequence ambiguity (Non-commutivity error) occurs due to non-zero sampling interval for digitization.**
- **To minimize this error source, we should sample gyro outputs as fast as possible.**
- **For this purpose, we need an attitude algorithm that is numerically efficient and stable**

-> "quaternions"

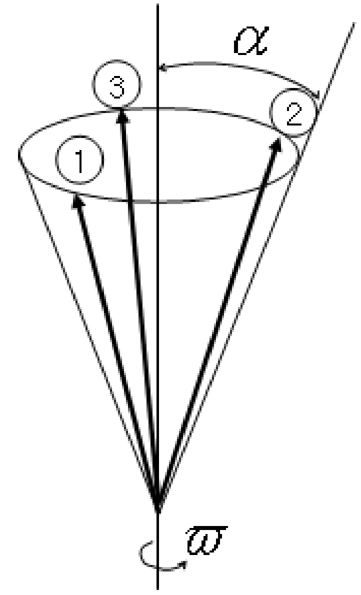
Attitude Algorithm Structure



- At each submajor interval, **gyro outputs** are sampled
- At each minor interval, **quaternions** are updated
- At each major interval, **transformation matrices** are updated

Coning algorithm ?

- **Among the various attitude dynamics, conning motions stimulate largest non-commutivity errors.**
- **The analysis of attitude error under the conning motion is very important in SDINS since it is the major environmental error source.**



2-Interval Coning Algorithm

- **incremental angle from gyro output**

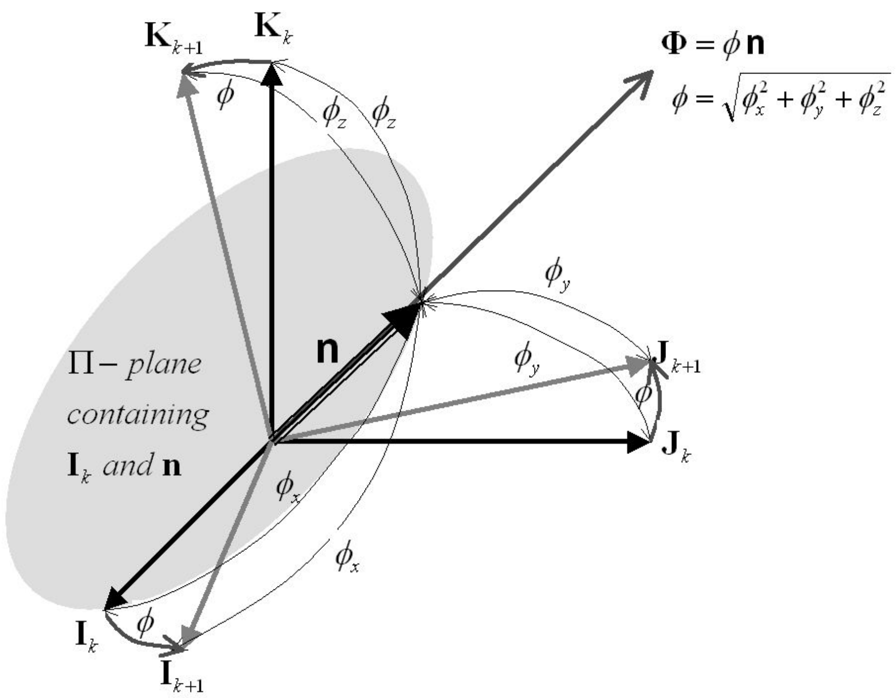
$$\theta_1 := \int_{\tau=T}^{T+h/2} \omega(\tau)d\tau \qquad \theta_2 := \int_{\tau=T+h/2}^{T+h} \omega(\tau)d\tau$$

- **(incremental) rotation vector**

$$\Phi = \theta_1 + \theta_2 + \frac{2}{3}\langle\theta_1\rangle\theta_2 = \begin{bmatrix} \phi_x \\ \phi_y \\ \phi_z \end{bmatrix}$$

- **(incremental) quaternion**

$$Q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ \hdashline Q_{sub} \end{bmatrix} := \begin{bmatrix} \cos(\phi/2) \\ \sin(\phi/2)\cos(\phi_x) \\ \sin(\phi/2)\cos(\phi_y) \\ \sin(\phi/2)\cos(\phi_z) \end{bmatrix}$$



- **Quaternion update by incremental quaternion**

$$Q_{k+1} = Q \otimes Q_k$$

(by quaternion multiplication)

updated (total) quaternion | incremental quaternion | previous (total) quaternion

$$= \Pi(Q)\ Q_k$$

(by matrix vector multiplication)

where

$$\Pi(Q) := \begin{bmatrix} q_0 & -Q_{sub}^T \\ Q_{sub} & q_0 I_{3\times 3} + \langle Q_{sub} \rangle \end{bmatrix} \qquad Q = \begin{bmatrix} q_0 \\ Q_{sub} \end{bmatrix}$$

Transformation Matrix by Quaternion

$$C_b^n = \begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) \\ 2(q_0q_3+q_1q_2) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_0q_1+q_2q_3) & q_0^2-q_1^2-q_2^2+q_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi-\cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi+\sin\phi\sin\psi \\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi+\cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi-\sin\phi\cos\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}$$

$$\phi = \arctan\left(\frac{c_{23}}{c_{33}}\right)$$

$$\psi = \arctan\left(\frac{c_{12}}{c_{11}}\right)$$

$$\theta = \arctan\left(\frac{-c_{13}}{\sqrt{1-c_{13}^2}}\right)$$

$$q_0 = C_{\phi/2}C_{\theta/2}C_{\psi/2} + S_{\phi/2}S_{\theta/2}S_{\psi/2}$$

$$q_1 = S_{\phi/2}C_{\theta/2}C_{\psi/2} - C_{\phi/2}S_{\theta/2}S_{\psi/2}$$

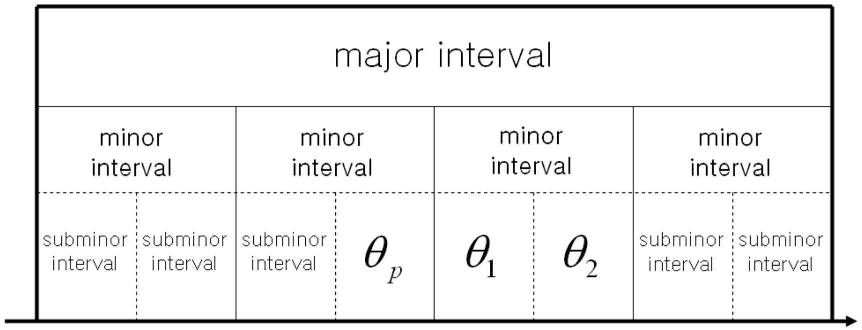
$$q_2 = C_{\phi/2}S_{\theta/2}C_{\psi/2} + S_{\phi/2}C_{\theta/2}S_{\psi/2}$$

$$q_3 = C_{\phi/2}C_{\theta/2}S_{\psi/2} - S_{\phi/2}S_{\theta/2}C_{\psi/2}$$

- By applying similar procedure, one can also get 3-, 4- and 5-sample algorithms.
- 3-sample coning algorithm is the most popular method for practical implementation.

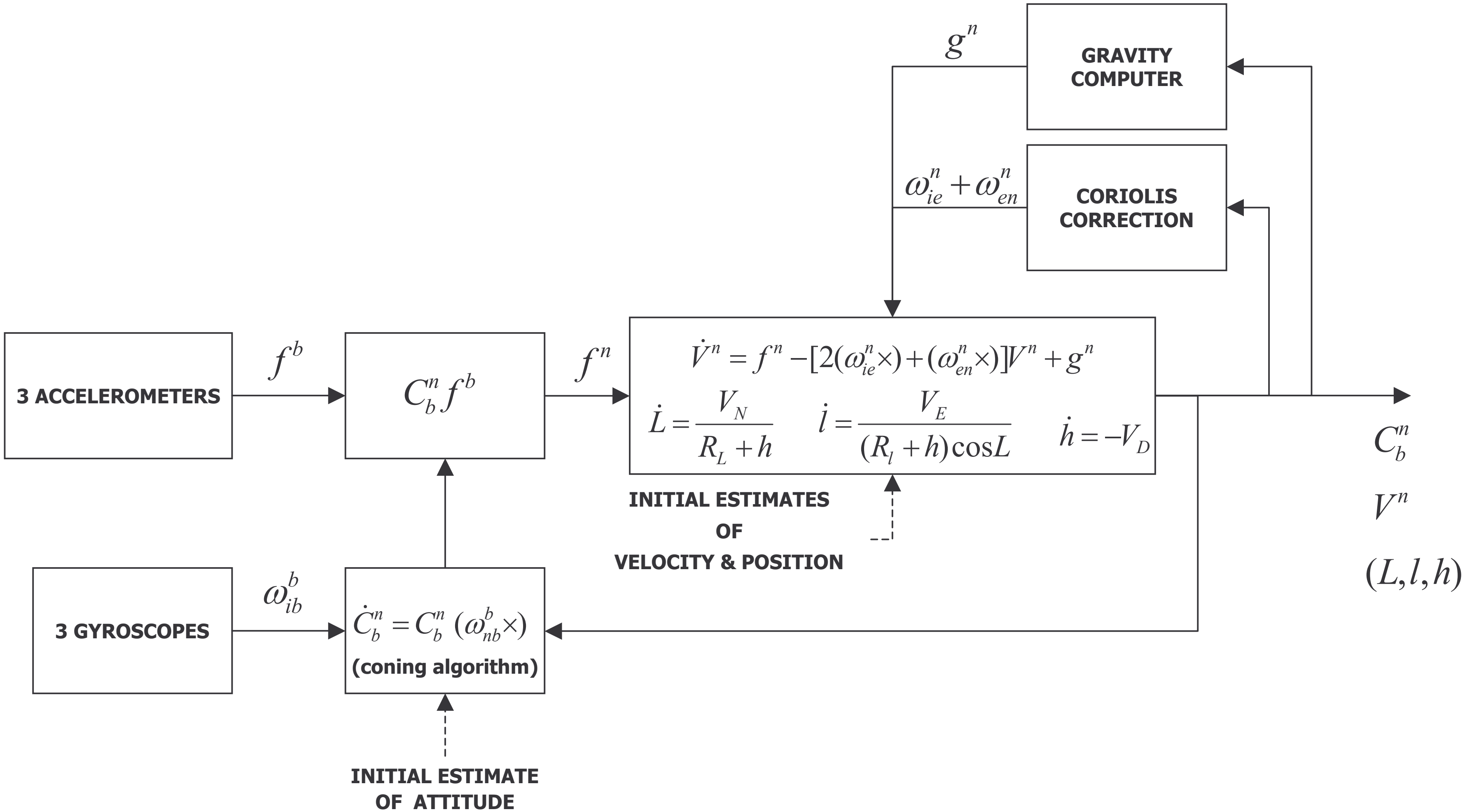
$$\Phi = \theta_1 + \theta_2 + \theta_3 + 0.4125\langle\theta_1\rangle\theta_3 + 0.7125\langle\theta_2\rangle(\theta_3 - \theta_1)$$

- There is also a branched method utilizing not only the angles in the same minor interval but also the fraction of angles sampled in the previous minor interval, i.e.



Summary of SDINS Algorithm

항법 및 정보시스템 연구실
Navigation & Information Systems Laboratory
항공전자 및 정보통신공학부
한국항공대학교
HANKUK AVIATION UNIVERSITY



GRAVITY COMPUTER
CORIOLIS CORRECTION
g^n
$\omega_{ie}^n + \omega_{en}^n$
3 ACCELEROMETERS
f^b
$C_b^n f^b$
f^n
$\dot{V}^n = f^n - [2(\omega_{ie}^n \times) + (\omega_{en}^n \times)]V^n + g^n$
$\dot{L} = \frac{V_N}{R_L + h}$ $\dot{l} = \frac{V_E}{(R_l + h)\cos L}$ $\dot{h} = -V_D$
INITIAL ESTIMATES OF VELOCITY & POSITION
3 GYROSCOPES
ω_{ib}^b
$\dot{C}_b^n = C_b^n (\omega_{nb}^b \times)$
(coning algorithm)
INITIAL ESTIMATE OF ATTITUDE
C_b^n
V^n
(L, l, h)

Error Modeling

항법 및 정보시스템 연구실
Navigation & Information Systems Laboratory
항공전자 및 정보통신공학부
한국항공대학교
HANKUK AVIATION UNIVERSITY

Perturbation Method 1

- Let

$$y = F(x, a, b, u)$$

then

$$\delta y = \delta F = \frac{\partial F}{\partial x}\delta x + \frac{\partial F}{\partial a}\delta a + \frac{\partial F}{\partial b}\delta b + \frac{\partial F}{\partial u}\delta u$$

Perturbation Method 2

- Consider the following dynamic equation

$$\dot{x} = ax + bu \quad \text{(p-1)}$$

where a, x, b, and u denote true values of interest.

- In actual situation, we never know the true values shown in Eq. (p-1).

However, we can utilize the estimates $\hat{a}$, $\hat{x}$, $\hat{b}$, and $\hat{u}$ instead of the true values a, x, b, and u as follows.

$$\dot{\hat{x}} = \hat{a}\hat{x} + \hat{b}\hat{u} \quad \text{(p-2)}$$

where

$$\hat{x} := x + \delta x,\ \hat{a} := a + \delta a,\ \hat{b} := b + \delta b,\ \hat{u} := u + \delta u, \quad \text{(p-3)}$$

and δa, δx, δb, and δu denote error terms.

- Substituting (p-3) to (p-1) and subtracting (p-2), we obtain

$$\delta\dot{x} = \hat{a}\,\delta x + \hat{b}\,\delta u + \delta a\hat{x} + \delta b\,\hat{u}$$

where products of errors are neglected.

Example: function

$$R_L = \frac{R_0(1-e^2)}{(1-e^2\sin^2 L)^{3/2}}, \qquad \delta R_L = R_{LL}\delta L$$

$$R_{LL} := \frac{\partial R_L}{\partial L} = -\frac{3}{2}\frac{R_0(1-e^2)}{(1-e^2\sin^2 L)^{5/2}}\frac{\partial}{\partial L}\left(-e^2\sin^2 L\right)$$

$$= -\frac{3}{2}\frac{R_0(1-e^2)}{(1-e^2\sin^2 L)^{5/2}}\left(-2e^2\sin L\cos L\right)$$

$$= \frac{3R_0(1-e^2)e^2\sin L\cos L}{(1-e^2\sin^2 L)^{5/2}}$$

Example: (error) time propagation

$$\dot{L} = \frac{V_N}{R_L + h} \qquad \delta\dot{L} = \frac{\partial \dot{L}}{\partial R_L}\delta R_L + \frac{\partial \dot{L}}{\partial h}\delta h + \frac{\partial \dot{L}}{\partial V_N}\delta V_N$$

$$\frac{\partial \dot{L}}{\partial R_L} = -\frac{V_N}{(R_L + h)^2} = \frac{1}{(R_L + h)}\frac{(-V_N)}{(R_L + h)} = \frac{\rho_E}{R_L + h}$$

$$\frac{\partial \dot{L}}{\partial V_N} = \frac{1}{R_L + h}$$

$$\frac{\partial \dot{L}}{\partial h} = -\frac{V_N}{(R_L + h)^2} = \frac{1}{(R_L + h)}\frac{(-V_N)}{(R_L + h)} = \frac{\rho_E}{R_L + h}$$

Example: (indirect) meas. eq.

$$\tilde{\rho}^{j} = \left(e_u^j\right)^T \left(R^j - X^e\right) + cb_{clock} + v_\rho^j$$

$$\hat{\rho}^{j} = \left(e_u^j\right)^T \left(R^j - \hat{X}^e\right) - c\hat{b}_{clock}$$

$$\hat{X}^e = X^e + \delta X^e$$

$$\hat{b}_{clock} = b_{clock} + \delta b_{clock}$$

$$z_\rho^j = \tilde{\rho}_u^j - \hat{\rho}_u^j = \left(e_u^j\right)^T \delta X^e - c\delta b_{clock} + v_\rho^j$$

$$\dot{X}_{INS} = F_{INS} X_{INS} + W_{INS}$$

where

$$X_{INS} = \begin{bmatrix} X_f^T & X_a^T \end{bmatrix}^T$$

$$X_f = \begin{bmatrix} \delta L & \delta l & \delta h & \delta V_N & \delta V_E & \delta V_D & \hat{\Phi}_N & \hat{\Phi}_E & \hat{\Phi}_D \end{bmatrix}^T$$

$$X_a = \begin{bmatrix} \nabla_X & \nabla_Y & \nabla_Z & \varepsilon_X & \varepsilon_Y & \varepsilon_Z \end{bmatrix}^T$$

$$W_{INS} = \begin{bmatrix} O_{1\times 3} & w_{aX} & w_{aY} & w_{aZ} & w_{gX} & w_{gY} & w_{gZ} & O_{1\times 6} \end{bmatrix}^T \sim N(O_{15\times 1}, Q_{INS})$$

$$F_{INS} = \begin{bmatrix} F_{11} & F_{12} & O_{3\times 3} & O_{3\times 3} & O_{3\times 3} \\ F_{21} & F_{22} & F_{23} & F_{24} & O_{3\times 3} \\ F_{31} & F_{32} & F_{33} & O_{3\times 3} & F_{35} \\ O_{3\times 3} & O_{3\times 3} & O_{3\times 3} & O_{3\times 3} & O_{3\times 3} \\ O_{3\times 3} & O_{3\times 3} & O_{3\times 3} & O_{3\times 3} & O_{3\times 3} \end{bmatrix}$$

$$F_{11} = \begin{bmatrix} \dfrac{R_{LL}\rho_E}{R_L + h} & 0 & \dfrac{\rho_E}{R_L + h} \\ \dfrac{\rho_N}{\cos L}\left(\tan L - \dfrac{R_{lL}}{R_l + h}\right) & 0 & -\dfrac{\rho_N \sec L}{R_l + h} \\ 0 & 0 & 0 \end{bmatrix} \quad F_{12} = \begin{bmatrix} \dfrac{1}{R_L + h} & & 0 \\ 0 & \dfrac{\sec L}{R_l + h} & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$F_{21} = \begin{bmatrix} \dfrac{\rho_E R_{LL}}{R_L + h} V_D - \left(\rho_N \sec^2 L + 2\Omega_N\right) V_E - \rho_N \rho_D R_{lL} & 0 & \dfrac{\rho_E}{R_L + h} V_D - \rho_N \rho_D \\ \left(2\Omega_N + \rho_N \sec^2 L + \dfrac{\rho_D R_{lL}}{R_l + h}\right) V_N - \left(\dfrac{\rho_N R_{lL}}{R_l + h} - 2\Omega_D\right) V_D & 0 & \dfrac{\rho_D}{R_l + h} V_N - \dfrac{\rho_N}{R_l + h} V_D \\ \rho_N^2 R_{lL} + \rho_E^2 R_{LL} - 2\Omega_D V_E & 0 & \rho_N^2 + \rho_E^2 \end{bmatrix}$$

$$F_{22} = \begin{bmatrix} \dfrac{V_D}{R_L + h} & 2\rho_D + 2\Omega_D & -\rho_E \\ -\rho_D - 2\Omega_D & \dfrac{V_N \tan L + V_D}{R_l + h} & 2\Omega_N + \rho_N \\ 2\rho_E & -2\Omega_N - 2\rho_N & 0 \end{bmatrix}, \quad F_{23} = \left\langle C_b^n f^b \right\rangle, \quad F_{24} = C_b^n$$

$$F_{31} = \begin{bmatrix} \Omega_D - \dfrac{\rho_N R_{lL}}{R_l + h} & 0 & \dfrac{-\rho_N}{R_l + h} \\ \dfrac{-\rho_E R_{LL}}{R_L + h} & 0 & \dfrac{-\rho_E}{R_L + h} \\ -\Omega_N - \rho_N \sec^2 L - \dfrac{\rho_D R_{lL}}{R_l + h} & 0 & \dfrac{-\rho_D}{R_l + h} \end{bmatrix}$$

$$F_{32} = \begin{bmatrix} 0 & \dfrac{1}{R_l + h} & 0 \\ \dfrac{-1}{R_L + h} & 0 & 0 \\ 0 & \dfrac{-\tan L}{R_l + h} & 0 \end{bmatrix} \quad F_{33} = \begin{bmatrix} 0 & \Omega_D + \rho_D & -\rho_E \\ -\Omega_D - \rho_D & 0 & \Omega_N + \rho_N \\ \rho_E & -\Omega_N - \rho_N & 0 \end{bmatrix}, \quad F_{35} = -C_b^n$$