§ Useful Quaternion Formulae

$$\Omega_1(\omega) := \begin{bmatrix} 0 & -(\omega)^T \\ \omega & -\langle\omega\rangle \end{bmatrix} \quad \text{(q-1)}$$

$$\Omega_2(\omega) := \begin{bmatrix} 0 & -(\omega)^T \\ \omega & \langle\omega\rangle \end{bmatrix} \quad \text{(q-2)}$$

$$Y(Q) := \begin{bmatrix} -Q_{sub}^T \\ q_0 I_{3\times3} - \langle Q_{sub}\rangle \end{bmatrix} \quad \text{(q-3)}$$

$$U(Q) := \begin{bmatrix} -Q_{sub}^T \\ q_0 I_{3\times3} + \langle Q_{sub}\rangle \end{bmatrix} \quad \text{(q-4)}$$

$$Q^T Q = 1 \quad \text{(q-5)}$$

$$U^T U = Y^T Y = I_{3\times3} \quad \text{(q-6)}$$

$$UU^T = YY^T = I_{4\times4} - QQ^T \quad \text{(q-7)}$$

$$Y^T Q = U^T Q = O_{3\times1} \quad \text{(q-8)}$$

$$Y^T \Omega_2 Q = \omega, \; U^T \Omega_1 Q = \omega \quad \text{(q-9)}$$

$$C_b^r = (Y_b^r)^T U_b^r, \; C_r^b = (U_b^r)^T Y_b^r \quad \text{(q-10)}$$

$$\langle Q_{sub}\rangle\langle\omega\rangle\langle Q_{sub}\rangle + \langle Q_{sub}\rangle\omega Q_{sub}^T + Q_{sub}\omega^T\langle Q_{sub}\rangle = -Q_{sub}^T Q_{sub}\langle\omega\rangle \quad \text{(q-11)}$$

$$Y^T \Omega_2 Y = \langle\omega\rangle, \; U^T \Omega_1 U = -\langle\omega\rangle \quad \text{(q-12)}$$

$$\Omega_1 - U\omega Q^T = \Omega_1 YY^T, \quad \Omega_2 - Y\omega Q^T = \Omega_2 YY^T \quad \text{(q-13)}$$

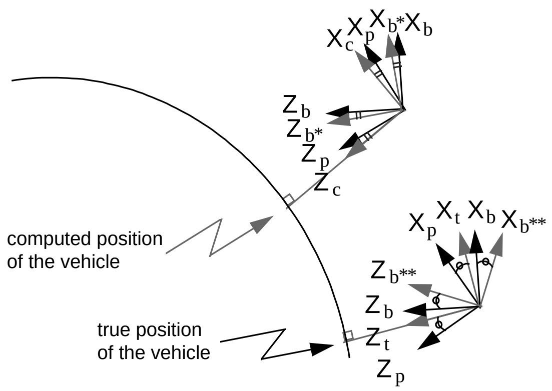
$$\dot{Y}_b^r = \frac{1}{2}\Omega_1(\omega_{ib}^b)Y_b^r + \frac{1}{2}\Omega_2(\omega_{ir}^r)Y_b^r + Q_b^r[\omega_{ir}^r]^T \quad \text{(q-14)}$$

$$\dot{U}_b^r = -\frac{1}{2}\Omega_1(\omega_{ib}^b)U_b^r - \frac{1}{2}\Omega_2(\omega_{ir}^r)Y_b^r + Q_b^r[\omega_{ib}^b]^T \quad \text{(q-15)}$$

$$Q \otimes S = \Pi_2(Q)S = \Pi_1(S)Q \quad \text{(q-16)}$$

$$\Pi_1(Q) := \begin{bmatrix} q_0 & -Q_{sub}^T \\ Q_{sub} & q_0 I_{3\times3} - \langle Q_{sub}\rangle \end{bmatrix}, \; \Pi_2(Q) := \begin{bmatrix} q_0 & -Q_{sub}^T \\ Q_{sub} & q_0 I_{3\times3} + \langle Q_{sub}\rangle \end{bmatrix} \quad \text{(q-17)}$$

§ Difference between True-Frame and Computer-Frame



SDINS computes the attitude of the vehicle and the velocity of the vehicle is obtained by integrating the following Coriolis force equation.

$$\dot{V} = f - \left\langle 2\omega_{ie} + \omega_{er} \right\rangle V + g \qquad \text{(f-1)}$$

where the subscript r represents a reference frame. The r-frame should be selected for the calculation of Eq. (f-1), and usually either the t-frame or the c-frame is used in this place. The configuration of the t-frame (or n-frame), c-frame and the p-frame used in the analysis of the terrestrial INS is depicted in Fig. 1. The t-frame (or n-frame) is the local level north pointing coordinate system defined on the true position of the vehicle, while the c-frame is the local level north pointing coordinate system defined on the erroneous calculated position of the vehicle.

§ True-Frame Velocity and Attitude Error Equation

The differential equation of the quaternion $Q_b^n (= Q_b^t)$ representing the rotation from the b-frame to the n-frame(t-frame) is as follows.

$$\begin{aligned}\dot{Q}_b^n &= \frac{1}{2}\Omega_1 Q_b^n - \frac{1}{2}\Omega_2 Q_b^n \\ &= \frac{1}{2}U_b^n \omega_{ib}^b - \frac{1}{2}Y_b^n \omega_{in}^n\end{aligned} \quad \text{(t-1)}$$

where $\Omega_1 \triangleq \Omega_1(\omega_{ib}^b)$ and $\Omega_2 \triangleq \Omega_2(\omega_{in}^n)$ are utilized for notational brevity. During the attitude calculation, we use the following equation instead of Eq. (t-1).

$$\begin{aligned}\dot{\hat{Q}}_b^n &= \frac{1}{2}\hat{\Omega}_1 \hat{Q}_b^n - \frac{1}{2}\hat{\Omega}_2 \hat{Q}_b^n \\ &= \frac{1}{2}\hat{U}_b^n \hat{\omega}_{ib}^b - \frac{1}{2}\hat{Y}_b^n \hat{\omega}_{in}^n\end{aligned} \quad \text{(t-2)}$$

where

$$\begin{aligned}&\hat{Q}_b^n = Q_b^n + \delta Q_b^n = Q_b^p, \quad \hat{\omega}_{ib}^b = \omega_{ib}^b + \varepsilon, \quad \hat{\omega}_{in}^n = \omega_{in}^n + \delta\omega_{in}^n \\ &\hat{U}_b^n = U(\hat{Q}_b^n) = U_b^n + \delta U_b^n, \quad \hat{Y}_b^n = Y(\hat{Q}_b^n) = Y_b^n + \delta Y_b^n \\ &\delta U_b^n = U(\delta Q_b^n), \quad \delta Y_b^n = Y(\delta Q_b^n) \\ &\hat{\Omega}_1 = \Omega_1 + \delta\Omega_1, \; \hat{\Omega}_2 = \Omega_2 + \delta\Omega_2\end{aligned} \quad \text{(t-3)}$$

By Eq. (t-1) and Eq. (t-2), the following error differential equation is obtained.

$$\delta\dot{Q}_b^n = \frac{1}{2}\Omega_1 \delta Q_b^n - \frac{1}{2}\Omega_2 \delta Q_b^n + \frac{1}{2}U_b^n \varepsilon - \frac{1}{2}Y_b^n \delta\omega_{in}^n \quad \text{(t-4)}$$

The SDINS velocity differential equation which is referenced by the c-frame can be written as

$$\dot{V}^n = C_b^n f^b - \langle 2\omega_{ie}^n + \omega_{en}^n \rangle V^n + g^n \quad \text{(t-5)}$$

Instead of Eq. (t-5), we use the following equation during the velocity calculation.

$$\dot{\hat{V}}^n = \hat{C}_b^n \hat{f}^b - \langle 2\hat{\omega}_{ie}^n + \hat{\omega}_{en}^n \rangle \hat{V}^n + \hat{g}^n \quad \text{(t-6)}$$

Subtracting Eq. (t-5) from Eq. (t-6), we obtain

$$\delta\dot{V}^n = \delta C_b^n f^b - \langle 2\omega_{ie}^n + \omega_{et}^n \rangle \delta V^n + \langle V^n \rangle (2\delta\omega_{ie}^n + \delta\omega_{en}^n) + C_b^n \nabla + \delta g^n \quad \text{(t-7)}$$

where

$$\begin{aligned}&\hat{V}^n = V^n + \delta V^n, \quad \hat{f}^b = f^b + \nabla, \quad \hat{C}_b^n = C_b^n + \delta C_b^n = C_b^p \\ &\hat{\omega}_{ie}^n = \omega_{ie}^n + \delta\omega_{ie}^n, \quad \hat{\omega}_{en}^n = \omega_{en}^n + \delta\omega_{en}^n, \quad \hat{g}^n = g^n + \delta g^n \\ &Q_n^p = Q_I + \delta Q_b^n \otimes Q_n^b\end{aligned} \quad \text{(t-8)}$$

In Eq. (t-8), δC_b^n which is used as the attitude error driving term satisfies

$$\delta C_b^n = (C_n^p - I) C_b^n \quad \text{(t-9)}$$

The transformation matrix C_n^p shown above satisfies the following relationship.

$$C_n^p = T(Q_n^p) = T(Q_I + \delta Q_b^n \otimes Q_n^b) \quad \text{(t-10)}$$

The transformation matrix which is constructed by the sum of the unit quaternion and the quaternion error can be expressed as follows. (For detailed derivation, see Appendix)

$$T(Q_I + \delta Q_b^n \otimes Q_n^b) = [1 + 2(Q_b^n)^T \delta Q_b^n] I + 2\langle (Y_b^n)^T \delta Q_b^n \rangle \quad \text{(t-11)}$$

By assuming that the calculated quaternion $\hat{Q}_b^n$ is normalized, the following equation holds.

$$(Q_b^n)^T \delta Q_b^n = 0 \tag{t-12}$$

By Eqs. (t-10), (t-11) and (t-12), the transformation matrix C_n^p satisfies the following relation.

$$C_n^p = I + 2\left\langle (Y_b^n)^T \delta Q_b^n \right\rangle \tag{t-13}$$

Using Eq. (t-9) and Eq. (t-13), the attitude error driving term $\delta C_b^n f^b$ in Eq. (t-7) is expressed as follows.

$$\begin{aligned} \delta C_b^n f^b &= 2\left\langle (Y_b^n)^T \delta Q_b^n \right\rangle f^n \\ &= -2\left\langle f^n \right\rangle (Y_b^n)^T \delta Q_b^n \end{aligned} \tag{t-14}$$

Using the above relation, we can rewrite Eq. (t-7) into the following equation.

$$\delta \dot{V}^n = -2\left\langle f^n \right\rangle (Y_b^n)^T \delta Q_b^n - \left\langle 2\omega_{ie}^n + \omega_{et}^n \right\rangle \delta V^n + \left\langle V^n \right\rangle \left(2\delta\omega_{ie}^n + \delta\omega_{en}^n\right) + C_b^n \nabla + \delta g^n \tag{t-15}$$

The quaternion error model which consists of Eq. (t-3) and Eq. (t-15) will be notated as the δQ_b^n error model. It is apparent that the calculation of $-2(Y_b^n)^T \delta Q_b^n$ in the first term $-2\left\langle f^n \right\rangle (Y_b^n)^T \delta Q_b^n$ of Eq. (t-15) requires more computation than the other terms. As shown in Eq. (t-13), the actual meaning of this term is the infinitesimal rotation between the n-frame and the p-frame. Thus, we can reduce the computation by defining this term as the equivalent tilt angle.

$$\widehat{\Phi}^n \underline{\underline{\Delta}} -2(Y_b^n)^T \delta Q_b^n \tag{t-16}$$

By Eq. (t-13) and Eq. (t-16), the defined $\widehat{\Phi}^n$ satisfies

$$C_n^p = I - \left\langle \widehat{\Phi}^n \right\rangle \tag{t-17}$$

Substituting Eq. (t-14) and Eq. (t-16) into Eq. (t-15), the velocity error differential equation is simplified as

$$\delta \dot{V}^n = \left\langle f^n \right\rangle \widehat{\Phi}^n - \left\langle 2\omega_{ie}^n + \omega_{et}^n \right\rangle \delta V^n + \left\langle V^n \right\rangle \left(2\delta\omega_{ie}^n + \delta\omega_{en}^n\right) + C_b^n \nabla + \delta g^n \tag{t-18}$$

In order to calculate the propagation of the equivalent tilt angle $\widehat{\Phi}^n$ in Eq. (t-18), we need its differential equation. The differentiation of $\widehat{\Phi}^n$ can be partitioned into the following form.

$$\dot{\widehat{\Phi}}^n = -2\left[(\dot{Y}_b^n)^T \delta Q_b^n + (Y_b^n)^T \delta \dot{Q}_b^n\right] \tag{t-19}$$

Y_b^n in Eq. (t-19) satisfies

$$Y_b^n (Y_b^n)^T = I - Q_b^n (Q_b^n)^T \tag{t-20}$$

By differentiating both sides of Eq. (t-20) and rearranging, the derivative of the Y_b^n is given by

$$\dot{Y}_b^n = \frac{1}{2}\left(\Omega_1 + \Omega_2\right) Y_b^n - Q_b^n (\omega_{in}^n)^T \tag{t-21}$$

Substituting Eq. (t-3) and Eq. (t-21) into Eq. (t-19) , we obtain

$$\begin{aligned} -2(\dot{Y}_b^n)^T \delta Q_b^n &= (Y_b^n)^T \Omega_2 \delta Q_b^n + (Y_b^n)^T \Omega_1 \delta Q_b^n - 2\omega_{in}^n (Q_b^n)^T \delta Q_b^n \\ -2(Y_b^n)^T \delta \dot{Q}_b^n &= -(Y_b^n)^T \Omega_1 \delta Q_b^n + (Y_b^n)^T \Omega_2 \delta Q_b^n - (Y_b^n)^T U_b^n \varepsilon + (Y_b^n)^T Y_b^n \delta\omega_{in}^n \\ \dot{\widehat{\Phi}}^n &= 2(Y_b^n)^T \Omega_2 \delta Q_b^n - (Y_b^n)^T U_b^n \varepsilon + (Y_b^n)^T Y_b^n \delta\omega_{in}^n - 2\omega_{in}^n (Q_b^n)^T \delta Q_b^n \end{aligned} \tag{t-22}$$

Using Eq. (t-20) and the relation of $Y_b^n \omega_{in}^n = \Omega_2 Q_b^n$, we can express the $(Y_b^n)^T \Omega_2$ in the first term of Eq. (t-22) as follows

$$(Y_b^n)^T \Omega_2 = (Y_b^n)^T \Omega_2 Y_b^n (Y_b^n)^T + \omega_{in}^n (Q_b^n)^T \tag{t-23}$$

Substituting Eq. (t-23) into Eq. (t-22) and using the relations of

$$(Y_b^n)^T \Omega_2 \left(\omega_{in}^n\right) Y_b^n = \left\langle \omega_{in}^n \right\rangle \tag{t-24}$$

$$(Y_b^n)^T U_b^n = C_b^n \tag{t-25}$$

$$(Y_b^n)^T Y_b^n = I_{3\times3} \tag{t-26}$$

the differential equation of the $\widehat{\Phi}^n$ is reduced to

$$\dot{\widehat{\Phi}}^n = -\langle \omega_{in}^n \rangle \widehat{\Phi}^n - C_b^n \varepsilon + \delta\omega_{in}^n \tag{t-27}$$

Both Eq. (t-18) and Eq. (t-27) constitute the SDINS $\widehat{\Phi}^n$ model. By applying Y_b^n to both sides of Eq. (t-16) from the left and using Eq. (t-12) and Eq. (t-20), the conversion equation from $\widehat{\Phi}^n$ to the quaternion error δQ_b^n is derived as

$$\delta Q_b^n = -\frac{1}{2} Y_b^n \widehat{\Phi}^n \tag{t-28}$$

§ Computer-Frame Velocity and Attitude Error Equation

The differential equation of the quaternion Q_b^c representing the rotation from the b-frame to the c-frame is as follows.

$$\begin{aligned}\dot{Q}_b^c &= \frac{1}{2}\Omega_1 Q_b^c - \frac{1}{2}\Omega_2 Q_b^c \\ &= \frac{1}{2}U_b^c \omega_{ib}^b - \frac{1}{2}Y_b^c \omega_{ic}^c\end{aligned} \tag{c-1}$$

where $\Omega_1 \triangleq \Omega_1\left(\omega_{ib}^b\right)$ and $\Omega_2 \triangleq \Omega_2\left(\omega_{ic}^c\right)$ are utilized for notational convenience.

During the attitude calculation, we use the following equation instead of Eq. (c-1).

$$\begin{aligned}\dot{\hat{Q}}_b^c &= \frac{1}{2}\hat{\Omega}_1 \hat{Q}_b^c - \frac{1}{2}\hat{\Omega}_2 \hat{Q}_b^c \\ &= \frac{1}{2}\hat{U}_b^c \hat{\omega}_{ib}^b - \frac{1}{2}\hat{Y}_b^c \hat{\omega}_{ic}^c\end{aligned} \tag{c-2}$$

By Eq. (c-1) and Eq. (c-2), the following error differential equation is obtained.

$$\delta\dot{Q}_b^c = \frac{1}{2}\Omega_1 \delta Q_b^c - \frac{1}{2}\Omega_2 \delta Q_b^c + \frac{1}{2}U_b^c \varepsilon \tag{c-3}$$

where

$$\begin{aligned}&\hat{Q}_b^c = Q_b^c + \delta Q_b^c = Q_b^p, \quad \hat{\omega}_{ib}^b = \omega_{ib}^b + \varepsilon, \quad \hat{\omega}_{ic}^c = \omega_{ic}^c \\ &\hat{U}_b^c = U_b^c + \delta U_b^c, \quad \hat{Y}_b^c = Y_b^c + \delta Y_b^c \\ &\hat{\Omega}_1 = \Omega_1 + \delta\Omega_1, \quad \hat{\Omega}_2 = \Omega_2 + \delta\Omega_2\end{aligned} \tag{c-4}$$

The SDINS velocity differential equation which is referenced by the c-frame can be written as

$$\dot{V}^c = C_b^c f^b - \left\langle 2\omega_{ie}^c + \omega_{ec}^c \right\rangle V^c + g^c \tag{c-5}$$

Instead of Eq. (c-5), we use the following equation during the velocity calculation.

$$\dot{\hat{V}}^c = \hat{C}_b^c \hat{f}^b - \left\langle 2\hat{\omega}_{ie}^c + \hat{\omega}_{ec}^c \right\rangle \hat{V}^c + \hat{g}^c \tag{c-6}$$

Subtracting Eq. (c-5) from Eq. (c-6), we obtain

$$\delta\dot{V}^c = \delta C_b^c f^b + C_b^c \nabla - \left\langle 2\omega_{ie}^c + \omega_{ec}^c \right\rangle \delta V^c + \delta g^c \tag{c-7}$$

where

$$\begin{aligned}&\hat{V}^c = V^c + \delta V^c, \quad \hat{f}^b = f^b + \nabla, \quad \hat{C}_b^c = C_b^c + \delta C_b^c = C_b^p \\ &\hat{\omega}_{ie}^c = \omega_{ie}^c, \quad \hat{\omega}_{ec}^c = \omega_{ec}^c, \quad \hat{g}^c = g^c + \delta g^c \\ &Q_c^p = Q_I + \delta Q_b^c \otimes Q_c^b\end{aligned} \tag{c-8}$$

In Eq. (c-8), δC_b^c which is used as the attitude error driving term satisfies

$$\delta C_b^c = \left(C_c^p - I\right) C_b^c \tag{c-9}$$

The transformation matrix C_c^p shown above satisfies the following relationship.

$$C_c^p = T\left(Q_c^p\right) = T\left(Q_I + \delta Q_b^c \otimes Q_c^b\right) \tag{c-10}$$

The transformation matrix which is constructed by the sum of the unit quaternion and the quaternion error can be expressed as follows.

$$T\left(Q_I + \delta Q_b^c \otimes Q_c^b\right) = \left(1 + 2Q_b^{c^T} \delta Q_b^c\right) I + 2\left\langle Y_b^{c^T} \delta Q_b^c \right\rangle \tag{c-11}$$

By assuming that the calculated quaternion $\hat{Q}_b^c$ is normalized, the following equation holds.

$$Q_b^{c^T} \delta Q_b^c = 0 \tag{c-12}$$

By Eqs. (c-10), (c-11) and (c-12), the transformation matrix C_c^p satisfies the following relation.

$$C_c^p = I + 2\left\langle Y_b^{c^T} \delta Q_b^c \right\rangle \tag{c-13}$$

Using Eq. (c-9) and Eq. (c-13), the attitude error driving term $\delta C_b^c f^b$ in Eq. (c-7) is expressed as follows.

$$\begin{aligned} \delta C_b^c f^b &= 2\left\langle Y_b^{c^T} \delta Q_b^c \right\rangle f^c \\ &= -2\left\langle f^c \right\rangle Y_b^{c^T} \delta Q_b^c \end{aligned} \tag{c-14}$$

Using the above relation, we can rewrite Eq. (c-7) into the following equation.

$$\delta \dot{V}^c = -2\left\langle f^c \right\rangle Y_b^{c^T} \delta Q_b^c + C_b^c \nabla - \left\langle 2\omega_{ie}^c + \omega_{ec}^c \right\rangle \delta V^c + \delta g^c \tag{c-15}$$

The quaternion error model which consists of Eq. (c-3) and Eq. (c-15) will be notated as the δQ_b^c error model. It is apparent that the calculation of $-2Y_b^{c^T} \delta Q_b^c$ in the first term $-2\left\langle f^c \right\rangle Y_b^{c^T} \delta Q_b^c$ of Eq. (c-15) requires more computation than the other terms. As shown in Eq. (c-13), the actual meaning of this term is the infinitesimal rotation between the c-frame and the p-frame. Thus, we can reduce the computation by defining this term as the equivalent tilt angle.

$$\hat{\Psi}^c \triangleq -2Y_b^{c^T} \delta Q_b^c \tag{c-16}$$

By Eq. (c-13) and Eq. (c-16), the defined $\hat{\Psi}^c$ satisfies

$$C_c^p = I - \left\langle \hat{\Psi}^c \right\rangle \tag{c-17}$$

Substituting Eq. (c-14) and Eq. (c-16) into Eq. (c-15), the velocity error differential equation is simplified as

$$\delta \dot{V}^c = \left\langle f^c \right\rangle \hat{\Psi}^c + C_b^c \nabla - \left\langle 2\omega_{ie}^c + \omega_{ec}^c \right\rangle \delta V^c + \delta g^c \tag{c-18}$$

In order to calculate the propagation of the equivalent tilt angle $\hat{\Psi}^c$ in Eq. (c-18), we need its differential equation. The differentiation of $\hat{\Psi}^c$ can be partitioned into the following form.

$$\dot{\hat{\Psi}}^c = -2\left[(\dot{Y}_b^c)^T \delta Q_b^c + (Y_b^c)^T \delta \dot{Q}_b^c \right] \tag{c-19}$$

Y_b^c in Eq. (c-19) satisfies

$$Y_b^c (Y_b^c)^T = I - (Q_b^c)(Q_b^c)^T \tag{c-20}$$

By differentiating both sides of Eq. (c-20) and rearranging, the derivative of the Y_b^c is given by

$$\dot{Y}_b^c = \frac{1}{2}(\Omega_1 + \Omega_2) Y_b^c - Q_b^c (\omega_{ic}^c)^T \tag{c-21}$$

Substituting Eq. (c-3) and Eq. (c-21) into Eq. (c-19) , we obtain

$$\dot{\hat{\Psi}}^c = 2(Y_b^c)^T \Omega_2 \delta Q_b^c - (Y_b^c)^T U_b^c \varepsilon - 2\omega_{ic}^c (Q_b^c)^T \delta Q_b^c \tag{c-22}$$

Using Eq. (c-20) and the relation of $Y_b^c \omega_{ic}^c = \Omega_2 Q_b^c$, we can express the $(Y_b^c)^T \Omega_2$ in the first term of Eq. (c-22) as follows

$$(Y_b^c)^T \Omega_2 = (Y_b^c)^T \Omega_2 Y_b^c (Y_b^c)^T + \omega_{ic}^c (Q_b^c)^T \tag{c-23}$$

Substituting Eq. (c-23) into Eq. (c-22) and using the relations of

$$(Y_b^c)^T \Omega_2 Y_b^c = \left\langle \omega_{ic}^c \right\rangle \tag{c-24}$$

$$(Y_b^c)^T U_b^c = C_b^c \tag{c-25}$$

the differential equation of the $\hat{\Psi}^c$ is reduced to

$$\dot{\hat{\Psi}}^c = -\left\langle \omega_{ic}^c \right\rangle \hat{\Psi}^c - C_b^c \varepsilon \tag{c-26}$$

Both Eq. (c-18) and Eq. (c-26) constitute the SDINS $\hat{\Psi}^c$ model, where **the attitude error equation is decoupled from the position and velocity error equations**. By applying Y_b^c to both sides of Eq. (c-16) from the left and using Eq. (c-12) and Eq. (c-20), the conversion equation from $\hat{\Psi}^c$ to the quaternion error δQ_b^c is derived as

$$\delta Q_b^c = -\frac{1}{2} Y_b^c \hat{\Psi}^c \tag{c-27}$$

$\widehat{\Psi}^c$ model	$\widehat{\Psi}^c = -2\hat{Y}_b^{c\,T}\delta Q_b^c$ $\delta\dot{V}^c = \left\langle \hat{f}^c \right\rangle \widehat{\Psi}^c + \hat{C}_b^c\nabla - \left\langle 2\hat{\omega}_{ie}^c + \hat{\omega}_{ec}^c \right\rangle \delta V^c + \delta g^c$ $\dot{\widehat{\Psi}}^c = -\left\langle \hat{\omega}_{ic}^c \right\rangle \widehat{\Psi}^c - \hat{C}_b^c\varepsilon$ $\delta Q_b^c = -\frac{1}{2}\hat{Y}_b^c\widehat{\Psi}^c$
$\widehat{\Phi}^n$ model	$\widehat{\Phi}^n = -2(\hat{Y}_b^n)^T\delta Q_b^n$ $\delta\dot{V}^n = \left\langle \hat{f}^n \right\rangle \widehat{\Phi}^n + \hat{C}_b^n\nabla - \left\langle 2\delta\omega_{ie}^n + \delta\omega_{en}^n \right\rangle \hat{V}^n - \left\langle 2\hat{\omega}_{ie}^n + \hat{\omega}_{en}^n \right\rangle \delta V^n + \delta g^n$ $\dot{\widehat{\Phi}}^n = -\left\langle \hat{\omega}_{in}^n \right\rangle \widehat{\Phi}^n - \hat{C}_b^n\varepsilon + \delta\omega_{in}^n$ $\delta Q_b^n = -\frac{1}{2}\hat{Y}_b^n\widehat{\Phi}^n$