§ Useful Quaternion Formulae

\[ \Omega_1 (\omega) := \begin{bmatrix} 0 & -\omega^T \\ \omega & -\omega \end{bmatrix} \]  
(q-1)

\[ \Omega_2 (\omega) := \begin{bmatrix} 0 & -\omega^T \\ \omega & \omega \end{bmatrix} \]  
(q-2)

\[ Y(Q) := \frac{-Q_{sub}^T}{q_0 I_{3x3} - \{Q_{sub}\}} \]  
(q-3)

\[ U(Q) := \frac{-Q_{sub}^T}{q_0 I_{3x3} + \{Q_{sub}\}} \]  
(q-4)

\[ Q^T Q = 1 \]  
(q-5)

\[ U^T U = Y^T Y = I_{3x3} \]  
(q-6)

\[ UU^T = YY^T = I_{4x4} - QQ^T \]  
(q-7)

\[ Y^T Q = U^T Q = O_{3x1} \]  
(q-8)

\[ Y^T \Omega_2 Q = \omega, \quad U^T \Omega_2 Q = \omega \]  
(q-9)

\[ C'_b = (Y_b)^T U'_b, \quad C''_b = (U'_b)^T Y'_b \]  
(q-10)

\[ \{Q_{sub}\} \omega \{Q_{sub}\} + \{Q_{sub}\} \omega Q_{sub}^T + Q_{sub} \omega^T \{Q_{sub}\} = -Q_{sub}^T Q_{sub} \{\omega\} \]  
(q-11)

\[ Y^T \Omega_2 Y = \{\omega\}, \quad U^T \Omega_2 U = -\{\omega\} \]  
(q-12)

\[ \Omega_1 - U \omega Q^T = \Omega_1 YY^T, \quad \Omega_2 - Y \omega Q^T = \Omega_2 YY^T \]  
(q-13)

\[ \dot{Y}'_b = \frac{1}{2} \Omega_1 (\omega'_b) Y'_b + \frac{1}{2} \Omega_2 (\omega'_b) Y'_b + Q'_b [\omega'_b]^T \]  
(q-14)

\[ \dot{U}'_b = -\frac{1}{2} \Omega_1 (\omega'_b) U'_b - \frac{1}{2} \Omega_2 (\omega'_b) Y'_b + Q'_b [\omega'_b]^T \]  
(q-15)

\[ Q \otimes S = \Pi_1 (Q) S = \Pi_1 (S) Q \]  
(q-16)

\[ \Pi_1 (Q) := \begin{bmatrix} q_0 & -Q_{sub}^T \\ Q_{sub} & q_0 I_{3x3} - \{Q_{sub}\} \end{bmatrix}, \quad \Pi_2 (Q) := \begin{bmatrix} q_0 & -Q_{sub}^T \\ Q_{sub} & q_0 I_{3x3} + \{Q_{sub}\} \end{bmatrix} \]  
(q-17)
§ Difference between True-Frame and Computer-Frame

SDINS computes the attitude of the vehicle and the velocity of the vehicle is obtained by integrating the following Coriolis force equation.

\[
\dot{V} = f - \left( 2\omega_e + \omega_e \right) V + g
\]  

where the subscript \( r \) represents a reference frame. The r-frame should be selected for the calculation of Eq. (f-1), and usually either the t-frame or the c-frame is used in this place. The configuration of the t-frame (or n-frame), c-frame and the p-frame used in the analysis of the terrestrial INS is depicted in Fig. 1. The t-frame (or n-frame) is the local level north pointing coordinate system defined on the true position of the vehicle, while the c-frame is the local level north pointing coordinate system defined on the erroneous calculated position of the vehicle.
§ True-Frame Velocity and Attitude Error Equation

The differential equation of the quaternion $Q^n_b (= Q^t_b)$ representing the rotation from the b-frame to the n-frame(t-frame) is as follows.

$$
\dot{Q}^n_b = \frac{1}{2} \Omega_1 Q^i_b - \frac{1}{2} \Omega_2 Q^e_b
$$

$$
= \frac{1}{2} U^n_b \alpha_b - \frac{1}{2} Y^n_b \omega^n_m
$$

(t-1)

where $\Omega_1 \Delta \Omega_1 (\alpha^b_m)$ and $\Omega_2 \Delta \Omega_2 (\omega^m_n)$ are utilized for notational brevity. During the attitude calculation, we use the following equation instead of Eq. (t-1).

$$
\dot{Q}^n_b = \frac{1}{2} \dot{\Omega}_1 \dot{Q}^i_b - \frac{1}{2} \dot{\Omega}_2 \dot{Q}^e_b
$$

(t-2)

$$
\dot{U}^n_b = U^n_b \alpha_b + \dot{\epsilon}_b, \quad \dot{Y}^n_b = Y^n_b + \dot{\gamma}_b
$$

$$
\dot{\Omega}_1 = \Omega_1 + \Delta \Omega_1, \quad \dot{\Omega}_2 = \Omega_2 + \Delta \Omega_2
$$

(t-3)

By Eq. (t-1) and Eq. (t-2), the following error differential equation is obtained.

$$
\ddot{Q}^n_b = \frac{1}{2} \Omega_1 \ddot{Q}^i_b - \frac{1}{2} \Omega_2 \ddot{Q}^e_b - \frac{1}{2} U^n_b \epsilon - \frac{1}{2} Y^n_b \delta \omega^n_m
$$

(t-4)

The SDINS velocity differential equation which is referenced by the c-frame can be written as

$$
\dot{V}^n = C^c_b f^b - \langle 2 \omega^n_w + \omega^n_m \rangle \hat{V}^n + \hat{g}^n
$$

(t-5)

Instead of Eq. (t-5), we use the following equation during the velocity calculation.

$$
\dot{V}^n = \hat{C}_b \dot{f}^b - \langle 2 \omega^n_c + \omega^n_m \rangle \hat{V}^n + \hat{g}^n
$$

(t-6)

Subtracting Eq. (t-5) from (t-6), we obtain

$$
\hat{\hat{V}}^n = \hat{\partial} C^c_b f^b - \langle 2 \omega^n_c + \omega^n_m \rangle \hat{\partial} \hat{V}^n + \langle \hat{V}^n \rangle \langle 2 \delta \omega^n_w + \delta \omega^n_m \rangle + C^c_n \hat{\partial} \hat{V}^n + \hat{\delta} \hat{g}^n
$$

(t-7)

where

$$
\hat{V}^n = V^n + \hat{\partial} V^n, \quad \hat{f}^b = f^b + \hat{\partial} f^b, \quad \hat{\hat{C}}_b = C^c_b + \hat{\partial} C^c_b = C^p_b
$$

$$
\hat{\omega}^n_w = \omega^n_w + \delta \omega^n_w, \quad \hat{\omega}^n_m = \omega^n_m + \delta \omega^n_m, \quad \hat{\hat{g}}^n = g^n + \hat{\delta} g^n
$$

$$
\hat{Q}^n_b = Q_i + \hat{\partial} Q^i_b \otimes Q^i_n
$$

(t-8)

In Eq. (t-8), $\hat{\partial} C^c_b$ which is used as the attitude error driving term satisfies

$$
\hat{\partial} C^c_b = (C^c_p - I) C^c_b
$$

(t-9)

The transformation matrix $C^p_n$ shown above satisfies the following relationship.

$$
C^p_n = T(Q^p_i) = T(Q_i + \hat{\partial} Q^i_b \otimes Q^i_n)
$$

(t-10)

The transformation matrix which is constructed by the sum of the unit quaternion and the quaternion error can be expressed as follows. (For detailed derivation, see Appendix)

$$
T(Q_i + \hat{\partial} Q^i_b \otimes Q^i_n) = \left[ 1 + 2(Q^p_i)^T \hat{\partial} Q^p_i \right] f + 2 \left( Y^p_b \right)^T \hat{\partial} Q^p_i
$$

(t-11)

By assuming that the calculated quaternion $\hat{Q}^n_b$ is normalized, the following equation holds.
\[(Q_b^n)^T \dot{Q}_b^n = 0 \]  
(t-12)

By Eqs. (t-10), (t-11) and (t-12), the transformation matrix \( C_n^p \) satisfies the following relation.
\[ C_n^p = I + 2\left(Y_b^n)^T \dot{Q}_b^n\right) \]  
(t-13)

Using Eq. (t-9) and Eq. (t-13), the attitude error driving term \( \dot{C}_b^n f^b \) in Eq. (t-7) is expressed as follows.
\[ \dot{C}_b^n f^b = 2\left(Y_b^n)^T \dot{Q}_b^n\right) f^n \]
\[ = -2\left(f^n\right)(Y_b^n)^T \dot{Q}_b^n \]  
(t-14)

Using the above relation, we can rewrite Eq. (t-7) into the following equation.
\[ \ddot{V}^n = -2\left(f^n\right)(Y_b^n)^T \dot{Q}_b^n - \left(2\omega_{ie}^n + \omega_{ri}^n\right) \dot{V}^n + \left(\dot{Y}^n\right) \left(2\delta\omega_{ie}^n + \delta\omega_{ri}^n\right) + C_b^n \dot{\nabla} + \delta g^n \]  
(t-15)

The quaternion error model which consists of Eq. (t-3) and Eq. (t-15) will be notated as the \( \Delta Q_b^n \) error model. It is apparent that the calculation of \(-2\left(f^n\right)(Y_b^n)^T \dot{Q}_b^n\) in the first term \(-2\left(f^n\right)(Y_b^n)^T \dot{Q}_b^n\) of Eq. (t-15) requires more computation than the other terms. As shown in Eq. (t-13), the actual meaning of this term is the infinitesimal rotation between the \( n \)-frame and the \( p \)-frame. Thus, we can reduce the computation by defining this term as the equivalent tilt angle.
\[ \dot{\Phi}^n \Delta = -2\left(Y_b^n\right)^T \dot{Q}_b^n \]  
(t-16)

By Eq. (t-13) and Eq. (t-16), the defined \( \dot{\Phi}^n \) satisfies
\[ C_n^p = I - \left(\dot{\Phi}^n\right) \]  
(t-17)

Substituting Eq. (t-14) and Eq. (t-16) into Eq. (t-15), the velocity error differential equation is simplified as
\[ \dot{V}^n = \left(f^n\right)\dot{\Phi}^n - \left(2\omega_{ie}^n + \omega_{ri}^n\right) \dot{V}^n + \left(\dot{Y}^n\right) \left(2\delta\omega_{ie}^n + \delta\omega_{ri}^n\right) + C_b^n \dot{\nabla} + \delta g^n \]  
(t-18)

In order to calculate the propagation of the equivalent tilt angle \( \dot{\Phi}^n \) in Eq. (t-18), we need its differential equation. The differentiation of \( \dot{\Phi}^n \) can be partitioned into the following form.
\[ \dot{\Phi}^n = -2\left(Y_b^n\right)^T \dot{Q}_b^n \]  
(t-19)

\[ Y_b^n Y_b^n = (Y_b^n)^T I - Q_b^n \]  
(t-20)

By differentiating both sides of Eq. (t-20) and rearranging, the derivative of the \( Y_b^n \) is given by
\[ \dot{Y}_b^n = \frac{1}{2} \left(\Omega_1 + \Omega_2\right) Y_b^n - Q_b^n (\omega_m^n)^T \]  
(t-21)

Substituting Eq. (t-3) and Eq. (t-21) into Eq. (t-19), we obtain
\[-2(Y_b^n)^T \delta Q_b^n = (Y_b^n)^T \Omega_2 \delta Q_b^n + (Y_b^n)^T \Omega_1 \delta Q_b^n - 2\omega_m^n (Q_b^n)^T \delta Q_b^n \]
\[-2(Y_b^n)^T \delta Q_b^n = -(Y_b^n)^T \Omega_2 \delta Q_b^n + (Y_b^n)^T \Omega_1 \delta Q_b^n - (Y_b^n)^T U_b^n \epsilon + (Y_b^n)^T Y_b^n \delta \omega_m^n \]
\[ \dot{\Phi}^n = 2\left(Y_b^n\right)^T \Omega_2 \dot{Q}_b^n - \left(Y_b^n\right)^T U_b^n \epsilon + \left(Y_b^n\right)^T Y_b^n \delta \omega_m^n - 2\omega_m^n (Q_b^n)^T \dot{Q}_b^n \]  
(t-22)

Using Eq. (t-20) and the relation of \( Y_b^n \omega_m^n = \Omega_2 Q_b^n \), we can express the \( (Y_b^n)^T \Omega_2 \) in the first term of Eq. (t-22) as follows
\[ (Y_b^n)^T \Omega_2 = (Y_b^n)^T \Omega_2 Y_b^n (Y_b^n)^T + \omega_m^n (Q_b^n)^T \]  
(t-23)

Substituting Eq. (t-23) into Eq. (t-22) and using the relations of
\[ (Y_b^n)^T \Omega_2 \omega_m^n = \langle \omega_m^n \rangle \]
\[ (Y_b^n)^T U_b^n = C_b^n \]  
(t-24)
\begin{align*}
(Y_b^n)^T Y_b^n &= I_{3,3} 	ag{t-26} \\
\text{the differential equation of the } \Phi^n \text{ is reduced to} \\
\dot{\Phi}^n &= -\{\omega^n\} \Phi^n - C^n_b \dot{\epsilon} + \delta \omega^n 	ag{t-27} \\
\text{Both Eq. (t-18) and Eq. (t-27) constitute the SDINS } \Phi^n \text{ model. By applying } Y_b^n \text{ to both sides of Eq. (t-16) from the left and using Eq. (t-12) and Eq. (t-20), the conversion equation from } \Phi^n \text{ to the quaternion error } \delta Q^n_b \text{ is derived as} \\
\delta Q^n_b &= -\frac{1}{2} Y_b^n \Phi^n 	ag{t-28}
\end{align*}
§ Computer-Frame Velocity and Attitude Error Equation

The differential equation of the quaternion $Q_b^c$ representing the rotation from the b-frame to the c-frame is as follows.

$$
\dot{Q}_b^c = \frac{1}{2} \Omega_1 Q_b^c - \frac{1}{2} \Omega_2 Q_b^c
$$

$$
= \frac{1}{2} U_b^c \omega_b^c - \frac{1}{2} Y_b^c \omega_c^c
$$

where $\Omega_1 \Delta \Omega_1 (\omega_b^c)$ and $\Omega_2 \Delta \Omega_2 (\omega_c^c)$ are utilized for notational convenience.

During the attitude calculation, we use the following equation instead of Eq. (c-1).

$$
\dot{Q}_b^c = \frac{1}{2} \Omega_1 \dot{Q}_b^c - \frac{1}{2} \Omega_2 \dot{Q}_b^c
$$

(c-2)

By Eq. (c-1) and Eq. (c-2), the following error differential equation is obtained.

$$
\dot{Q}_b^c = \frac{1}{2} \Omega_1 \dot{Q}_b^c - \frac{1}{2} \Omega_2 \dot{Q}_b^c + \frac{1}{2} U_b^c \varepsilon
$$

(c-3)

where

$$
\dot{Q}_b^c = Q_b^c + \delta Q_b^c = Q_b^c, \quad \dot{\omega}_b^c = \omega_b^c + \varepsilon, \quad \dot{\omega}_c^c = \omega_c^c
$$

$$
U_b^c = U_b^c + \delta U_b^c, \quad Y_b^c = Y_b^c + \delta Y_b^c
$$

$$
\dot{\Omega}_1 = \Omega_1 + \delta \Omega_1, \quad \dot{\Omega}_2 = \Omega_2 + \delta \Omega_2
$$

(c-4)

The SDINS velocity differential equation which is referenced by the c-frame can be written as

$$
\dot{V}^c = C_b^c f^b \dot{h}^b - \left(2 \dot{\omega}_e^c + \dot{\omega}_c^c \right) V^c + g^c
$$

(c-5)

Instead of Eq. (c-5), we use the following equation during the velocity calculation.

$$
\dot{V}^c = \dot{C}_b^c f^b - \left(2 \dot{\omega}_e^c + \dot{\omega}_c^c \right) \dot{V}^c + \dot{g}^c
$$

(c-6)

Subtracting Eq. (c-5) from Eq. (c-6), we obtain

$$
\delta \dot{V}^c = \delta C_b^c f^b + C_b^c \delta V^b - \left(2 \dot{\omega}_e^c + \dot{\omega}_c^c \right) \delta V^c + \delta g^c
$$

(c-7)

where

$$
\dot{V}^c = V^c + \delta V^c, \quad \dot{f}^b = f^b + \delta f^b, \quad \dot{C}_b^c = C_b^c + \delta C_b^c = C_b^c
$$

$$
\dot{\omega}_e^c = \omega_e^c + \delta \omega_e^c, \quad \dot{\omega}_c^c = \omega_c^c + \delta \omega_c^c, \quad \dot{g}^c = g^c + \delta g^c
$$

$$
Q_b^c = Q_b^c + \delta Q_b^c \otimes Q_b^c
$$

(c-8)

In Eq. (c-8), $\delta C_b^c$ which is used as the attitude error driving term satisfies

$$
\delta C_b^c = (C_b^c - I) C_b^c
$$

(c-9)

The transformation matrix $C_b^c$ shown above satisfies the following relationship.

$$
C_b^c = T(Q_b^c) = T(Q_b^c + \delta Q_b^c \otimes Q_b^c)
$$

(c-10)

The transformation matrix which is constructed by the sum of the unit quaternion and the quaternion error can be expressed as follows.

$$
T(Q_b^c + \delta Q_b^c \otimes Q_b^c) = \left[1 + 2Q_b^c^T \delta Q_b^c \right] Q_b^c + 2 \left[ Q_b^c^T \delta Q_b^c \right] Q_b^c
$$

(c-11)

By assuming that the calculated quaternion $\hat{Q}_b^c$ is normalized, the following equation holds.
\[ Q_b^T \partial Q_b^c = 0 \]  
(c-12)

By Eqs. (c-10), (c-11) and (c-12), the transformation matrix \( C_c^p \) satisfies the following relation.

\[ C_c^p = I + 2 \left\{ Y_b^{c^T} \partial Q_b^c \right\} \]  
(c-13)

Using Eq. (c-9) and Eq. (c-13), the attitude error driving term \( \partial C_c^p f^b \) in Eq. (c-7) is expressed as follows.

\[ \partial C_c^p f^b = 2 \left\{ Y_b^{c^T} \partial Q_b^c \right\} f^c \]

\[ = -2 \left\{ f^c \right\} Y_b^{c^T} \partial Q_b^c \]  
(c-14)

Using the above relation, we can rewrite Eq. (c-7) into the following equation.

\[ \dot{\tilde{\Psi}}^c - 2 Y_b^{c^T} \partial \tilde{Q}_b^c \]  
(c-16)

By Eq. (c-13) and Eq. (c-16), the defined \( \tilde{\Psi}_c \) satisfies

\[ C_c^p = I - \left\{ \tilde{\Psi}_c \right\} \]  
(c-17)

Substituting Eq. (c-14) and Eq. (c-16) into Eq. (c-15), the velocity error differential equation is simplified as

\[ \dot{\tilde{V}}^c = \left\{ f^c \right\} \tilde{\Psi}_c + \left( Y_b^{c^T} \right) \partial \tilde{Q}_b^c + \left( Y_b^{c^T} \partial \tilde{Q}_b^c \right) \omega_c^b + g_c^b \]  
(c-18)

In order to calculate the propagation of the equivalent tilt angle \( \tilde{\Psi}_c \) in Eq. (c-18), we need its differential equation. The differentiation of \( \tilde{\Psi}_c \) can be partitioned into the following form.

\[ \hat{\dot{\tilde{\Psi}}}^c = -2 \left\{ \left( \dot{Y}_b^{c^T} \right) \partial \tilde{Q}_b^c + \left( Y_b^{c^T} \right) \partial \dot{\tilde{Q}}_b^c \right\} \]  
(c-19)

\[ Y_b^{c^T} \]  
(c-20)

By differentiating both sides of Eq. (c-20) and rearranging, the derivative of the \( Y_b^{c^T} \) is given by

\[ \dot{Y}_b^{c^T} = \frac{1}{2} \left( \Omega_1 + \Omega_2 \right) \dot{Y}_b^{c^T} - \dot{Q}_b^c \left( \omega_c^b \right)^T \]  
(c-21)

Substituting Eq. (c-3) and Eq. (c-21) into Eq. (c-19), we obtain

\[ \hat{\dot{\tilde{\Psi}}}^c = 2 \left\{ \left( \dot{Y}_b^{c^T} \right) \Omega_2 \partial \tilde{Q}_b^c - \left( Y_b^{c^T} \right) \partial \dot{\tilde{Q}}_b^c \right\} \omega_c^b \]  
(c-22)

Using Eq. (c-20) and the relation of \( Y_b^{c^T} \omega_c^b = \Omega_2 \dot{Q}_b^c \), we can express the \( \left( Y_b^{c^T} \right) \Omega_2 \) in the first term of Eq. (c-22) as follows

\[ \left( Y_b^{c^T} \right) \Omega_2 = \left( Y_b^{c^T} \right) \Omega_2 Y_b^{c^T} \left( Y_b^{c^T} \right) + \omega_c^b \left( Q_b^c \right)^T \]  
(c-23)

Substituting Eq. (c-23) into Eq. (c-22) and using the relations of

\[ \left( Y_b^{c^T} \right) \Omega_2 Y_b^{c^T} = \left( \omega_c^b \right)^T \]  
(c-24)

\[ \left( Y_b^{c^T} \right) \dot{U}_b^c = C_b^c \]  
(c-25)

the differential equation of the \( \tilde{\Psi}_c \) is reduced to
\[
\dot{\Psi}_c = -\left( \omega_c^c \right) \Psi_c^c - C_b^c \epsilon 
\]

Both Eq. (c-18) and Eq. (c-26) constitute the SDINS model, where the attitude error equation is decoupled from the position and velocity error equations. By applying \( Y_b^c \) to both sides of Eq. (c-16) from the left and using Eq. (c-12) and Eq. (c-20), the conversion equation from \( \Psi_c^c \) to the quaternion error \( \delta Q_b^c \) is derived as

\[
\delta Q_b^c = -\frac{1}{2} Y_b^c \Psi_c^c
\]
<table>
<thead>
<tr>
<th>Model</th>
<th>Equation</th>
</tr>
</thead>
</table>
| \( \Psi^c \) model | \[
\dot{\Psi}^c = -2\hat{Y}_{b}^{cT} \delta Q_b^c \\
\delta \dot{V}^c = \left( \hat{f}^c \right)^2 \dot{\Psi}^c + \hat{C}_b^c \nabla - \left( 2\hat{\omega}_c^c + \hat{\omega}_c^c \right) \nabla \Psi^c + \delta g^c \\
\dot{\Psi}^c = -\left( \delta \hat{\omega}_c^c \right) \Psi^c - \hat{C}_b^c \delta g^c \\
\delta Q_b^c = -\frac{1}{2} \hat{Y}_{b}^{cT} \Psi^c
\] |
| \( \Phi^a \) model | \[
\dot{\Phi}^a = -2\hat{Y}_{b}^{aT} \delta Q_b^a \\
\delta \dot{V}^a = \left( \hat{f}^a \right)^2 \dot{\Phi}^a + \hat{C}_b^a \nabla - \left( 2\hat{\omega}_c^a + \hat{\omega}_c^a \right) \nabla \Phi^a + \delta g^a \\
\dot{\Phi}^a = -\left( \delta \hat{\omega}_c^a \right) \Phi^a - \hat{C}_b^a \delta g^a \\
\delta Q_b^a = -\frac{1}{2} \hat{Y}_{b}^{aT} \Phi^a
\] |